The Calculation of the Probable Error from the Squares of the Adjusted Direct Observations of Equal Precision and Fechner's Formula

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Let λ denote the deviations of the observations from their arithmetic mean, let σ denote the mean error, and ρ the probable error. Then the optimal estimate of ρ is well known to be given by the following formulae,

$$\rho = 0.67449\ldots\sigma$$

$$\sigma = \sqrt{\frac{[\lambda\lambda]}{n-1}}\left[1 \pm \sqrt{\frac{1}{2(n-1)}}\right] \tag{1}$$

where the square root in the bracket is the man error in the estimate of $\hat{\sigma}$, expressed as a fraction of $\hat{\sigma}$. It is our intention to provide a somewhat more rigorous derivation of this formula u nder the Gaussian law of error than given elsewhere, even where the principles of probability theory are used.

If ϵ denotes a true error of an observation, then the future probability of a ser $\epsilon_1, \ldots, \epsilon_n$ is

$$\left[\frac{h}{\sqrt{\pi}}\right]^n e^{-h^2[\epsilon\epsilon]} d\epsilon_1 \ldots d\epsilon_n. \tag{3}$$

For given $\epsilon_1, \ldots, \epsilon_n$, by setting the probability of a hypothesis h proportional to this expresion, one obtains an optimal value of σ^2

$$\frac{1}{2h^2} = \hat{\sigma}^2 = \frac{[\epsilon\epsilon]}{n}. \tag{A}$$

However, since the ϵ are unknown, we are forced to estimate $[\epsilon\epsilon]$ and this may be regarded as a weakness of previous derivations. This deficiency may be removed by the consideration that a set $\lambda_1, \ldots, \lambda_n$ may arise from true errors in an infinity of ways. But since only the λ are given, we must calculate the future probability of a set $\lambda_1, \ldots, \lambda_n$ and take this expression as proportional to the probability of the hypothesis about h.

1 Probability of a Set $\lambda_1, \ldots, \lambda_n$ of Deviations from the Arithmetic Mean

In expression (3) we introduce the variables $\lambda_1, \ldots, \lambda_{n-1}$ and $\bar{\epsilon}$ in place of the ϵ by the equations:

$$\epsilon_1 = \lambda_1 + \bar{\epsilon}, \qquad \epsilon_2 = \lambda_2 + \bar{\epsilon}, \ldots$$
$$\epsilon_{n-1} = \lambda_{n-1} + \bar{\epsilon}, \qquad \epsilon_n = -\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1} + \bar{\epsilon}$$

This transformation is in accord with the known relations between the errors ϵ and deviations λ, since the addition of the equations gives $n\bar{\epsilon} = [\epsilon]$; at the same time the condition $[\lambda] = 0$ is satisfied. The determinant of the transformation, a determinant of the nth degree, is

$$\begin{vmatrix} 1 & \cdot & \cdot & & \cdot & 1 \\ \cdot & 1 & \cdot & & \cdot & 1 \\ \cdot & \cdot & 1 & & \cdot & 1 \\ \\ \cdot & \cdot & \cdot & & 1 & 1 \\ -1 & -1 & -1 & & -1 & 1 \end{vmatrix} = n.$$

Consequently expression (3) becomes

$$n\left[\frac{h}{\sqrt{\pi}}\right]^n e^{-h^2[\lambda\lambda]+h^2 n\bar{\epsilon}^2} d\lambda_1 d\lambda_2 \ldots d\lambda_{n-1} d\bar{\epsilon} \tag{B}$$

where $[\lambda\lambda] = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$; $\lambda_n = -\lambda_1 - \lambda_2 - \cdots - \lambda_{n-1}$. If we now integrate over all possible values of $\bar{\epsilon}$, we obtain for the probability of the set $\lambda_1 \ldots \lambda_n$ the expression

$$\sqrt{n}\left[\frac{h}{\sqrt{\pi}}\right]^{n-1} e^{-h^2[\lambda\lambda]} d\lambda_1 d\lambda_2 \ldots d\lambda_{n-1}. \tag{3}$$

This may be verified by integration over all possible values of $\lambda_1 \ldots \lambda_{n-1}$, which yields unity, as required.

2 Optimal Hypothesis on h for Given Deviations λ

For given values of the λ's we set the probability of a hypothesis on h proportional to expression (3). A standard argument then yields the optimal estimate of h as the value maximizing (3). Differentiation shows that this occurs when

$$\frac{1}{2h^2} = \frac{[\lambda\lambda]}{n-1}.$$

which establishes the first part of formula (1)*.

3 Probability of a Sum $[\lambda\lambda]$ of Squares of the Deviations λ

The probability that $[\lambda\lambda]$ lies between u and $u + du$ is from (3)

$$\sqrt{n}\left[\frac{h}{\sqrt{\pi}}\right]^{n-1} \int d\lambda_1 \ldots \int d\lambda_{n-1} e^{-h^2[\lambda\lambda]}, \tag{4}$$

*In the same way it is possible by strict use of probability theory to derive a formula for σ^2 when n observations depend on m unknowns, a result which the author has established to his satisfaction and will communicate elsewhere.

integrated over all $\lambda_1 \dots \lambda_{n-1}$ satisfying

$$u \leq [\lambda\lambda] \leq u + du.$$

We now introduce $n-1$ new variables t by means of the equations

$$\begin{aligned}
t_1 &= \sqrt{2}(\lambda_1 + \tfrac{1}{2}\lambda_2 + \tfrac{1}{2}\lambda_3 + \tfrac{1}{2}\lambda_3 + \cdots + \tfrac{1}{2}\lambda_{n-1}) \\
t_2 &= \qquad \sqrt{\frac{3}{2}}(\lambda_2 + \tfrac{1}{3}\lambda_3 + \tfrac{1}{3}\lambda_4 + \cdots + \tfrac{1}{3}\lambda_{n-1}) \\
t_3 &= \qquad\qquad \sqrt{\frac{4}{3}}(\lambda_3 + \tfrac{1}{4}\lambda_4 + \cdots + \tfrac{1}{4}\lambda_{n-1}) \\
. &= \quad . \qquad . \qquad . \\
t_{n-1} &= \qquad\qquad\qquad \sqrt{\frac{n}{n-1}}\lambda_{n-1}
\end{aligned}$$

With the determinant $\sqrt{n}$ of the transformation, the above expression becomes

$$\sqrt{n}\left[\frac{h}{\sqrt{\pi}}\right]^{n-1} \int dt_1 \dots \int dt_{n-1} e^{-h^2[tt]},$$

the limits of integration being determined by the condition

$$u \leq [tt] \leq u + du.$$

We now recognize that the probability for the sum of squares of the n deviations λ, $[\lambda\lambda] = u$, is precisely the same probability that the sum of squares $[tt]$ of $n-1$ true errors equals u. This last probability I gave in Schlömlich's journal, 1875, p. 303, according to which

$$\frac{h^{n-1}}{\Gamma(\frac{n-1}{2})} u^{\frac{n-3}{2}} u^{-h^2 u} du, \tag{5}$$

is the probability that the sum of squares $[\lambda\lambda]$ of the deviations λ of n equally precise observations from their mean lies between u and $u + du$. Integration of (5) from $u = 0$ to ∞ gives unity.

4 The Mean Error of the Formula $\hat{\sigma} = \sqrt{[\lambda\lambda] : (n-1)}$

Since it is difficult to obtain a generally valid formula for the probable error of this formula, we confine ourselves to the mean error.

The mean error of the formula $\hat{\sigma}^2 = \frac{[\lambda\lambda]}{n-1}$ is known exactly, namely $\sigma^2 \sqrt{2 : (n-1)}$. We have therefore

$$\hat{\sigma}^2 = \frac{[\lambda\lambda]}{n-1}\left[1 \pm \sqrt{\frac{1}{2(n-1)}}\right]$$

and if n is large it follows by a familiar argument that

$$\hat{\sigma} = \sqrt{\frac{[\lambda\lambda]}{n-1}}\left[1 \pm \frac{1}{2}\sqrt{\frac{1}{2(n-1)}}\right].$$

Formula (1) results. However, if n is small, for example equal to 2, this argument lacks all validity. For then $\sqrt{2:(n-1)}$ is no longer small compared to 1, in fact even larger than 1 for $n = 2$. We now proceed as follows.

The mean squared error of the formula

$$\hat{\sigma} = \sqrt{[\lambda\lambda]:(n-1)}$$

is the mean value of

$$\left[\sqrt{\frac{\lambda\lambda]}{n-1}} - \sigma\right]^2.$$

If one develops the square and recalls that $[\lambda\lambda]:(n-1)$ has mean σ^2 or $1:2h^2$, it follows that the mean of the above is

$$\frac{1}{h^2} - \frac{\sqrt{2}}{h}\left[\sqrt{\frac{[\lambda\lambda]}{n-1}}\right].$$

where the term in large brackets must be replaced by its mean value.

Consideration of formula (5) yields for the mean value of $\sqrt{[\lambda\lambda]}$ the expression

$$\frac{h^{n-1}}{\Gamma(\frac{n-1}{2})}\int_0^\infty u^{\frac{n-2}{2}} u^{-h^2u^2} du, \text{i.e.,} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})},$$

so that the mean squared error of $\hat{\sigma}$ is

$$\frac{1}{h^2}\left[1 - \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\sqrt{\frac{2}{n-1}}\right].$$

We must therefore regard the following formula as more accurate than (1):

$$\hat{\sigma} = \sqrt{\frac{[\lambda\lambda]}{n-1}}\left[1 \pm \surd\left\{2 - \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\sqrt{\frac{8}{n-1}}\right\}\right]$$
$$\hat{\rho} = 0.67449\ldots\hat{\sigma}, \tag{6}$$

where the square root following $\pm$ signifies the mean error of the formula for $\hat{\sigma}$.

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