Application of the Method of Least Squares
to the Elements of the Planet Pallas

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In volume I of the *Gottingen Nachrichten*, Gauss gave the application of his method to the correction of the elements of the planet Pallas. Since the illustrious mathematician developed the algorithm outlined more briefly in his great work *Theoria Motus Corporum Coelestium* (see the previous Note) on this example, we felt we should translate here this part of his memoire. Since the first part requires an extensive knowledge of the theory of planetary motion, we shall not reproduce it, and we take as starting point the twelve equations which the six elements of the orbit ought to satisfy.

Denoting these corrections by

$$dL, \quad dZ, \quad d\pi, \quad dp, \quad d\Omega, \quad di$$

the equations obtained by Gauss are the following

$$\begin{array}{llllllllll}
0 & = & - & 183.93'' & + & 0.79363 dL & + & 143.66 dZ & + & 0.39493 d\pi \\
 & & & & + & 0.95920 d\phi & - & 0.18856 d\Omega & + & 0.17387 di; \\
0 & = & - & 6.81'' & - & 0.02658 dL & + & 46.71 dZ & + & 0.02658 d\pi \\
 & & & & - & 0.20858 d\phi & + & 0.15946 d\Omega & + & 1.25782 di; \\
0 & = & - & 0.06'' & + & 0.58880 dL & + & 358.12 dZ & + & 0.26208 d\pi \\
 & & & & - & 0.85234 d\phi & + & 0.14912 d\Omega & + & 0.17775 di; \\
0 & = & - & 3.09'' & + & 0.01318 dL & + & 28.39 dZ & + & 0.01318 d\pi \\
 & & & & - & 0.07861 d\phi & + & 0.91704 d\Omega & + & 0.54365 di; \\
0 & = & - & 0.02'' & + & 1.73436 dL & + & 1846.17 dZ & - & 0.54603 d\pi \\
 & & & & - & 2.05662 d\phi & - & 0.18853 d\Omega & - & 0.17445 di; \\
0 & = & - & 8.98'' & - & 0.12606 dL & - & 227.42 dZ & + & 0.12606 d\pi \\
 & & & & - & 0.38939 d\phi & - & 0.17176 d\Omega & - & 1.35441 di; \\
0 & = & - & 2.31'' & + & 0.99584 dL & + & 1579.03 dZ & + & 0.06456 d\pi \\
 & & & & + & 1.99545 d\phi & - & 0.06040 d\Omega & - & 0.33750 di; \\
0 & = & + & 2.47'' & - & 0.08089 dL & - & 67.22 dZ & + & 0.08089 d\pi \\
 & & & & - & 0.09970 d\phi & - & 0.46350 d\Omega & + & 1.22803 di; \\
0 & = & + & 0.01'' & + & 0.65311 dL & + & 1329.09 dZ & + & 0.38994 d\pi \\
 & & & & - & 0.08439 d\phi & - & 0.04305 d\Omega & + & 0.34268 di; \\
0 & = & + & 38.12'' & - & 0.00218 dL & + & 38.47 dZ & + & 0.00218 d\pi \\
 & & & & - & 0.18710 d\phi & + & 0.47301 d\Omega & - & 1.14371 di; \\
0 & = & - & 317.73'' & + & 0.69957 dL & + & 1719.32 dZ & + & 0.12913 d\pi \\
 & & & & - & 1.38787 d\phi & + & 0.17130 d\Omega & - & 0.08360 di; \\
0 & = & + & 117.97'' & - & 0.01315 dL & - & 43.84 dZ & + & 0.01315 d\pi \\
 & & & & + & 0.02929 d\phi & + & 1.02138 d\Omega & - & 0.27187 di.
\end{array}$$

From the nature of the observations which furnished the tenth of these equations, it is judged to inspire too little confidence to make use of it, and the six unknowns will be determined only from the other eleven.

The following explanations are literally translated from Gauss's Memoire

2 (§13 in the original)

Since it is impossible for us to satisfy the eleven proposed equations exactly, that is to say, to make all the right hand sides zero, we shall seek to make the sum of their squares as small as possible.

One sees easily that if one considers the linear functions

$$\begin{array}{lllllllllllll}
n & + & ap & + & bq & + & cr & + & ds & + & \dots & = & w, \\
n' & + & a'p & + & b'q & + & c'r & + & d's & + & \dots & = & w', \\
n'' & + & a''p & + & b''p & + & c''r & + & d''s & + & \dots & = & w'', \\
\multicolumn{13}{c}{\dotfill}
\end{array}$$

the equations which must be solved in order to make

$$\Omega = w + w' + w'' + \dots$$

a minimum, are

$$\begin{array}{lllllll}
aw & + & a'w' & + & a''w'' & + & \dots = 0, \\
bw & + & b'w' & + & b''w'' & + & \dots = 0, \\
cw & + & c'w' & + & c''w'' & + & \dots = 0, \\
\multicolumn{7}{c}{\dotfill}
\end{array}$$

or, defining the following abbreviations

$$\begin{array}{lllllllll}
an & + & a'n' & + & a''n'' & + & \dots & = & (an), \\
a^2 & + & a'^2 & + & a''^2 & + & \dots & = & (aa), \\
ab & + & a'b' & + & a''b'' & + & \dots & = & (ab), \\
\multicolumn{9}{c}{\dotfill} \\
b^2 & + & b\prime 2 & + & b\prime\prime 2 & + & \dots & = & (bb), \\
bc & + & b'c' & + & b''c'' & + & \dots & = & (bc) \\
\multicolumn{9}{c}{\dotfill}
\end{array}$$

p, d, r, s, etc. should be determined by the following equations

$$\begin{array}{lllllllllll}
(an) & + & (aa)p & + & (ab)q & + & (ac)r & + & \dots & = & 0, \\
(bn) & + & (ab)p & + & (bb)q & + & (bc)r & + & \dots & = & 0, \\
(cn) & + & (ac)p & + & (bc)q & + & (cc)r & + & \dots & = & 0, \\
\multicolumn{11}{c}{\dotfill}
\end{array}$$

The process of solution, very tedious when the number of unknowns is considerable, can be simplified notably in the following way. Suppose that besides the coefficients (an), (aa), etc. (of which the number if $\frac{1}{2}(i^2 + 3i)$, if the number of unknowns is i) one has calculated the sum

$$n^2 + n'^2 + n''^2 + \dots = (nn);$$

one sees easily that one has

$$\begin{array}{rrlrlrlrlrl}
\Omega & = & & (nn) & + & 2(an)p & + & 2(bn)q & + & 2(cn)r & + \ \ldots \\
& & + & (aa)p^2 & + & 2(ab)pq & + & 2(ac)pr & + & \ldots & \\
& & + & (bb)q^2 & + & 2(bc)qr & + & 2(bd)qs & + & \ldots & \\
& & + & (cc)r^2 & + & 2(cd)rs & + & \ldots; & & &
\end{array}$$

and, denoting

$$(an) + (aa)p + (ap)q + ...$$

by A, all the terms of $\frac{A^2}{(aa)}$ which contain the factor p, are found in the expression Ω, and consequently

$$\Omega - \frac{A^2}{(aa)}$$

is a function independent of p. This is why, setting

$$\begin{array}{lcccl}
(na) & - & \frac{(an)^2}{(aa)} & = & (nn,1), \\
(ba) & - & \frac{(an)(bn)}{(aa)} & = & (bn,1), \\
(ca) & - & \frac{(an)(cn)}{(aa)} & = & (cn,1), \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(bb) & - & \frac{(ab)^2}{(aa)} & = & (bb,1), \\
(bc) & - & \frac{(ab)(ac)}{(aa)} & = & (bc,1), \\
(bd) & - & \frac{(ab)(ad)}{(aa)} & = & (bd,1) \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

one has

$$\begin{array}{rrclclclc}
- \frac{A^2}{(aa)} & = & (aa,1) & + & 2(bn,1)q & + & 2(cn,1)r & + & 2(dn,1)s \ \ldots \\
& & & + & (bb,1)q^2 & + & 2(bc,1)qr & + & 2(bd,1)qs \ \ldots \\
& & & & & + & (cc,1)r^2 & + & 2(cd,1)rs \ \ldots \\
& & & & & & & + & \ldots
\end{array}$$

We shall denote this function by Ω'.

Similarly, setting

$$(bn,1) \quad + \quad (bb,1)q \quad + \quad (bc,1)r \quad + \quad \ldots \quad = \quad B,$$

the difference

$$\Omega' - \frac{B^2}{(bb,1)}$$

will be independent of q; we shall represent it by Ω''.

Similarly, setting

$$\begin{array}{lcccl}
(nn,1) & - & \frac{(bn,1)^2}{(bb,1)} & = & (nn,2), \\
(cn,1) & - & \frac{(bn,1)(bc,1)}{(bb,1)} & = & (cn,2), \\
(cc,1) & - & \frac{(bc,1)^2}{(bb,2)} & = & (cc,2), \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(cn,2) & + & (cc,2)r & + & (cd,2)s \quad + \quad \ldots \quad = \quad C
\end{array}$$

the difference

$$-\frac{C^2}{(cc,2)}$$

will be a function independent of r.

Continuing in this way we shall form a sequence of expressions , ', ", etc., of which the last will be independent of the various unknowns and is denoted by (nn,), if denotes the number of these unknowns; then we shall have

$$\Omega = \frac{A^2}{(aa)} + \frac{B^2}{(bb,1)} + \frac{C^2}{(cc,2)} + \frac{D^2}{(dd,3)} + \cdots + (nn,\mu)$$

One can easily prove that since is a sum of squares

$$w^2 + w'^2 + w''^2 + \ldots,$$

and cannot become negative, the denominators (aa), $(bb,1)$, $(cc,2)$, etc. are all positive. (For brevity we omit the details of the proof.) Accordingly, the minimum value of obviously corresponds to the values of the unknowns for which

$$A = 0, \quad B = 0, \quad C = 0, \quad ...,$$

and, by starting to solve the system with the last equation, which contains only one of them, one finds the values of p, q, r, s, etc. without having to carry out any elimination. The method gives at the same time the minimum value of , which is (nn,).

3 (§14 in the original)

Let us apply these principles to our example, in which p, q, r, s, etc. are replaced by dL, dZ, $d\pi$, $d\phi$, $d\Omega$, di. By careful calculation I have found

(nn)	$=$	$+148848$	(ac)	$=$	-0.09344	(cc)	$=$	$+0.71917$
(an)	$=$	-371.09	(ad)	$=$	-2.28516	(cd)	$=$	$+1.13382$
(bn)	$=$	-580104	(ae)	$=$	-0.34664	(ce)	$=$	$+0.06400$
(cn)	$=$	-113.45	(af)	$=$	-0.18194	(cf)	$=$	$+0.26341$
(dn)	$=$	$+268.53$	(bb)	$=$	$+10834225$	(dd)	$=$	$+12.00340$
(en)	$=$	$+94.26$	(bc)	$=$	-49.06	(de)	$=$	-0.37137
(fn)	$=$	-31.81	(bd)	$=$	-3229.77	(df)	$=$	-0.11762
(aa)	$=$	$+5.91569$	(be)	$=$	-198.64	(ee)	$=$	$+2.28215$
(ab)	$=$	$+7203.91$	(bf)	$=$	-143.05	(ef)	$=$	-0.36136
(ff)	$=$	-5.62456						

from which one obtains

$(nn,1)$	$=$	$+125569$	$(bc,1)$	$=$	$+62.13$	$(cf,1)$	$=$	$+0.26054$
$(bn,1)$	$=$	-138534	$(bd,1)$	$=$	-510.58	$(dd,1)$	$=$	$+11.12064$
$(cn,1)$	$=$	-119.31	$(be,1)$	$=$	$+213.84$	$(de,1)$	$=$	-0.50528
$(dn,1)$	$=$	-125.18	$(bf,1)$	$=$	$+73.45$	$(df,1)$	$=$	-0.18790
$(en,1)$	$=$	$+72.52$	$(cc,1)$	$=$	$+0.71769$	$(ee,1)$	$=$	$+2.26185$
$(fn,1)$	$=$	-43.22	$(cd,1)$	$=$	$+1.09773$	$(ef,1)$	$=$	-0.37202
$(bb,1)$	$=$	$+2458225$	$(ce,1)$	$=$	-0.05852	$(ff,1)$	$=$	$+5.61905$

Similarly

$$
\begin{array}{llllllll}
(nn,2) & = & +117763 & (cc,2) = & +0.71612 & (de,2) & = & -0.46088 \\
(cn,2) & = & -115.81 & (cd,2) = & +1.11063 & (df,2) & = & -0.17265 \\
(dn,2) & = & -153.95 & (ce,2) = & -0.06392 & (ee,2) & = & +2.24325 \\
(en,2) & = & +84.57 & (cf,2) = & +0.25868 & (ef,2) & = & -0.37841 \\
(fn,2) & = & -39.03 & (dd,2) = & +11.01466 & (ff,2) & = & +5.61686
\end{array}
$$

From which;

$$
\begin{array}{lllllllll}
(nn,3) & = & +99034 & (dd,3) & = & +9.29213 & (ee,3) & = & +2.23754 \\
(dn,3) & = & +25.66 & (de,3) & = & -0.36175 & (ef,3) & = & -0.35532 \\
(en,3) & = & +74.23 & (df,3) & = & -0.5738 & (ff,3) & = & +5.52342 \\
(fn,3) & = & +2.75 & & & & & &
\end{array}
$$

Similarly:

$$
\begin{array}{lllllllll}
(nn,4) & = & +98963 & (fn,4) & = & +4.33 & (ef,4) & = & -0.37766 \\
(en,5) & = & +75.23 & (ee,4) & = & +2.22346 & (ff,4) & = & +5.42383
\end{array}
$$

From which:

$$
\begin{array}{lllllllll}
(nn,5) & = & +96418 & (fn,5) & = & +17.11 & (ff,5) & = & +5.42383
\end{array}
$$

From which finally we obtain

$$(nn,6) = +96364$$

Thus we have the following six equations

$$
\begin{array}{lllllll}
0 & = & +17.11'' & +5.42383di & & & \\
0 & = & +75.23'' & +2.22346d\Omega & -0.37766di & & \\
0 & = & +25.66'' & +9.29213d\phi & -0.36175d\Omega & -0.57384di & \\
0 & = & -115.81'' & +0.71612d\pi & +1.11063d\phi & -0.06392d\Omega & +0.25868di \\
0 & = & -13854'' & +2458225dZ & +62.13d\pi & -0.51058d\phi & +213.84d\Omega \\
 & & +73.45di & & & & \\
0 & = & -371.09'' & +5.91569dL & +7203.91dZ & -0.00344d\pi & -2.20516d\phi \\
 & & -0.34664d\Omega & -0.18194di & & &
\end{array}
$$

from which one obtains;

$$
\begin{aligned}
di &= -3.15'' \\
d\Omega &= -34.37'' \\
d\phi &= -4.29'' \\
d\pi &= +166.44'' \\
dZ &= +0.054335'' \\
dL &= -3.06''
\end{aligned}
$$

These are the corrections which should be applied to the elements originally found for the planet.

Taken from *Work (1803–1826) on the Theory of Least Squares*, trans. H F Trotter, Technical Report No.5, Statistical Techniques Research Group, Princeton, NJ: Princeton University 1957.