On the Chronometric Determination of Longitudes
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Let θ, θ', θ'' etc. be the periods (n in number), at which a chronometer has determined the differences a, a', a'', etc. with the times of places whose longitudes are x, x', x'', etc. θ, θ', θ'', being supposed reduced to the time of a single place and u denoting the daily advances of the chronometer; one would have, if the instrument were perfectly regular, the equations

$$a - u - x = a' -' u - x' = a'' -'' u - x'' = \dots .$$

In order that these equations suffice for the determination of the unknowns x, x', x'', ..., u, it is necessary, for one thing, to consider one of the longitudes as given, and for another, it is necessary that at least two observations have been made in the same place, so that at least two of the unknowns x, x', x'', etc. are equal to each other. If among these quantities there are only two which are identical, the problem is completely determined; in the contrary case it becomes indeterminate, and one should proceed to satisfy the equations

$$\begin{array}{ccccccccccccc}
0 & = & a & - & a' & + & (\theta' & - & \theta)u & - & x & + & x' \\
0 & = & a' & - & a'' & + & (\theta'' & - & \theta')u & - & x' & + & x'' \\
0 & = & a'' & - & a''' & + & (\theta''' & - & \theta'')u & - & x'' & + & x''' \\
\multicolumn{13}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

as exactly as possible, from the inevitable imperfections of the chronometer will never permit all of them to be satisfied rigorously. However, one should not assign to these equations equal weight, for the quantities

$$\begin{array}{ccccccccccc}
a & - & a' & + & (\theta' & - & \theta)u & - & x & + & x' \\
a' & - & a'' & + & (\theta'' & - & \theta')u & - & x' & + & x'' \\
\multicolumn{11}{c}{\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

represent the accumulations of all the variations in the motion of the chronometer in the intervals $\theta' - \theta$, $\theta'' - \theta'$, etc. and if a good chronometer is involved to which one can truly attribute an average motion without a variation which keeps increasing in one directions, the average value to be expected for such a sum can be considered as proportional to the square root of the elapsed time.

Thus one should, in the application of the method of least squares, consider the preceding equations as having weights inversely proportional to the differences $\theta' - \theta$, $\theta'' - \theta'$, $\theta''' - \theta''$, etc.

The solution then offers no difficulty, and furnishes the most likely values of x, x', x'', etc. as well as the weight of each determination.

However I shall add several remarks.

I: If the first and last observation have been made at the same place, the most probable value of u is that which results from comparison of these extreme observations.

The calculations then become very simple, for by virtue of a theorem which is very easy to demonstrate, one may replace u in the equations by its most likely value, or, what comes to the same thing, on may use this value as if it were exact to correct the observations and to reduce them to those which would be made with a fictitious chronometer whose rate of gain was zero.

II. If one simply attributes to the various equations weights equal to

$$\frac{1}{\theta'' - \theta} \qquad \frac{1}{\theta'' - \theta'} \qquad \frac{1}{\theta''' - \theta''}$$

the unit of precision for the weights obtained will be the exactitude of that which one would obtain by the aid of the same chronometer observed only two times, and at one day's interval; but in order to compare the results obtained by the aid of various chronometers of unequal precision, on the greater or less perfection of such chronometer used.

To arrive as it I suppose that the expressions

$$\begin{array}{ccccccccccc} a & - & a' & + & (\theta' & - & \theta)u & - & x & + & x' \\ a' & - & a'' & + & (\theta'' & - & \theta')u & - & x' & + & x'' \\ \multicolumn{11}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

become λ, λ', λ'', etc. respectively, when one substitutes for the unknowns their most probable values. Let

$$\frac{\lambda^2}{\theta' - \theta} \quad + \quad \frac{\lambda\prime 2}{\theta'' - \theta'} \quad + \quad \frac{\lambda''^2}{\theta''' - \theta''} \quad + \cdots = S$$

if ν is the number of unknowns and one puts

$$m = \sqrt{\frac{S}{n - \nu - 1}}$$

the specific factor relating to each chronometer is proportional to $\frac{1}{m^2}$ or to $\frac{n-\nu-1}{S}$ and one can consider m as the deviation of the average motion which is to be expected during a day.

III: The preceding rules are relevant to a chronometer whose motion is not subject to any noticeable irregularity which increases with time. If this hypothesis were untenable one might assume, when the observations do not include an excessively long period, a variation in the daily gain of the instrument, proportional to the time thus producing an additional unknown.

The equations would then take the following form:

$$\begin{array}{ccccccccccccc} 0 & = & a & - & a' & + & (\theta'- & \theta)u & + & (\theta'^2- & \theta^2)v & - & x & + & x' \\ \multicolumn{15}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

IV: Concerning the solution of the equations according to the method of least squares, it is perhaps not unuseful to recollect that one should begin in most cases by calculating an approximate value for the unknowns, and the apply the method to the determination of the small corrections to which the values should be subjected.

It seemed useful to recall the general advice, because many calculators seem to have forgotten it and been led to calculations which were more laborious and perhaps less exact.

I have determined the behaviour of the following chronometers

				1		4		Breguet 3056		Barraud 904		Kassel 1252	
		h	′										
Greenwich	30 June	3	22	$-8'$	$17.14''$	$+1'$	$2.37''$						
	25 July	2	15	10	44.39	1	32.15	$+30'$	$59.75''$	$+48'$	$29.20''$	$+50'$	$29.31''$
	28 ″	3	13	11	0.69	1	36.96	30	50.07	48	40.24	50	39.69
	2 Aug.	1	15	11	28.48	1	44.44	30	31.78	48	58.87	50	52.14
	17 Aug.	10	18	12	59.40	2	6.24	29	35.69	49	57.83	51	38.66
	25 ″	7	27	13	47.98	2	15.84	29	10.48	50	27.15	52	2.45
	10 Sep.	7	40	15	24.47	2	40.36						
Helgoland	3 July	3	40	-40	8.00	-30	26.84						
	22 ″	12	40	42	2.02	30	3.89	-0	20.34	$+16$	47.39	$+18$	48.39
	5 Aug.	1	48	43	18.11	29	43.35	1	10.24	17	37.51	19	26.77
	11 ″	13	9	43	35.77	29	33.43	1	32.75	18	1.30	19	47.22
	30 ″	19	30	45	53.08	29	7.96	2	40.67	19	17.03	20	47.68
	6 Sep.	3	6	46	51.56	28	58.94	3	4.55	19	43.80	21	6.56
	7 ″	8	42	46	38.72	28	56.71						
Altona	6 Aug.	5	55	-51	38.95	-37	55.76	-9	28.50	$+9$	28.48	$+11$	16.25
	9 ″	12	35	51	57.35	37	50.03	9	38.81	9	40.30	11	27.76
	31 ″	9	57	54	10.33	37	21.30	10	56.68	11	5.92	12	25.96
	4 Sept.	22	12	54	39.16	37	15.21	11	15.36	11	24.49	12	48.10
Bremen	13 Aug.	0	2	-47	50.65	-33	16.49	-5	23.37	$+14$	21.86	$+16$	5.83

Let us for example take Breguet's chronometer 3056. Let zero be the longitude of Helogoland, -x that of Greenwich, y that of Altona. I do not take account here of that of Bremen, since having only one observation for this town, it is impossible to control it. I count the time from the first comparison of the Chronometer No. 1 (Greenwich June 30 3h 22m). Substituting for Breguet's chronometer a fictitious instrument with daily advance zero, we find

θ	θ	θ
$22.4 + 60.20''$	$37.1 - 434.98 \ + y$	$61.6 + 61.32$
$25.0 + 1949.60 - x$	$40.4 - 433.49 \ + y$	$62.2 - 432.53 + y$
$28.0 + 1950.87 - x$	$42.4 + 59.88$	$66.8 - 434.98 + y$
$32.9 + 1950.29 - x$	$48.3 + 1949.60 - x$	$68.0 + 60.19$
$35.9 + 59.08$	$56.2 + 1952.74 - x$	

In the equations above, the unknowns x and y are separated, which facilitates their determination; we find for x four determinations

	Weight
$1889.40''$	$\frac{1}{2{,}6} = 0.38$
1891.21	$\frac{1}{3{,}0} = 0.33$
1889.78	$\frac{1}{5{,}9} = 0.17$
1891.42	$\frac{1}{3.4} = 0.19$

from which one obtains

$$x = 1890.36'' \qquad 1.07$$

and similarly one finds

$$y = 494.12'' \qquad 3.83$$

According to these values, the fictitious chronometer would indicate, in Helgoland time

θ			λ	θ			λ	θ			λ
22.4	60.20”			37.1	59.14	$+$	0.06”	61.6	61.32	$-$	1.06
25.0	59.24	$-$	0.96	40.4	60.63	$+$	1.47	62.2	61.59	$-$	0.27
28.0	60.51	$+$	1.27	42.4	59.88	$-$	0.75	66.8	59.14	$-$	2.45
32.9	59.93	$-$	0.58	48.3	59.24	$-$	0.62	68.0	60.19	$+$	1.05
35.9	59.08	$-$	0.85	56.2	62.38	$+$	3.14				

from which one obtains

$$S = 6.00$$

$$m = \sqrt{\frac{6}{13-3}}$$

and the standard error to be expected is

$$\text{for } x \ 0.75'', \qquad \text{for } y \ 0.40''$$

The results furnished by the five chronometers give

				Standard error to be expected	Weight
Breguet	x	=	1890.36”	0.75	1.78
Kassel			1893.39	0.67	2.23
Barraud			1892.32	0.49	4.16
1			1892.39	0.43	5.41
4			1892.52	0.35	8.16
Average	x	=	1892.35		21.74

Similarly one finds for $y$lccccc

				Standard error to be expected	Weight
Breguet	y	=	494.12	0.10	6.25
Kassel			493.89	0.36	7.72
Barraud			493.67	0.21	14.79
1			493.98	0.29	11.89
4			494.16	0.24	17.36
					58.01

The number placed under the heading of weight in the last column is the reciprocal of the square of the standard error to be expected, taking as unit weight that which corresponds to observations giving a standard error to be expected of 1”, so that, for Altona, the standard error to be expected is $\sqrt{\frac{1}{58.01}} = 0.13''$; but it is preferable to consider the numbers in the last column as indicating merely ratios, and to deduce the absolute precision from the difference between the values of these final results found for x and y by means of each chronometer. The precision found in this way will be a little too large, since the determinations of time at Greenwich, at Helgoland and at Altona do not have an absolute precision, so that consequently whatever the number of chronometers, the errors arising from this source will always have some effect in each final result.

One may similarly, in the following way, obtain the longitude of Bremen.

Let z be this longitude to the east of Helgoland; the comparison of the Bremen chronometer gives the position of the fictitious chronometer as

$$-164.52'' + z$$

and one deduces from comparison with previous results

$$\begin{array}{ll} & \text{Weight} \\ z = 225.40'' & \frac{1}{1.4} = 0.7; \end{array}$$

the others give

$$\begin{array}{rr} z = 224.76 & \frac{1}{4.5} = 0.2 \\ 225.24 & 0.9 \end{array}$$

The weight 0.9 should be multiplied by $\frac{10}{6.000}$; the five chronometers give

Breguet	225.24	1.5
Kassel	225.84	1.9
Barraud	225.39	3.6
1	226.04	2.9
4	224.86	4.3
		14.2

The longitude of Bremen, which according to his would be $268.54''$ to the west of Altona, is naturally affected by errors in the determination of the time at Bremen, and this difference appears to be too small by several seconds. According to my triangulations, the tower of Anagarius is $273.51''$ of time to the west of Gottingen, and the observatory of Olbers $271.19''$.

Taken from *Work (1803–1826) on the Theory of Least Squares*, trans. H F Trotter, Technical Report No.5, Statistical Techniques Research Group, Princeton, NJ: Princeton University 1957.