Abraham de Moivre

IN interesting and valuable communications to *Biometrika*, of the final issues for 1924 and 1925, Karl Pearson set forth certain facts (not all new) which will doubtless result in Abraham de Moivre occupying a more important place than before in the history of mathematics. The results are reached by a careful study of (*a*) "James Bernoulli's Theorem, and (*b*) a publication of Moivre dated November 12, 1733. Regarding this publication Pearson makes certain statements which require comment, since from them wrong inferences might readily be drawn.

This publication is entitled: "Approximatio ad Summan Terminorum Binomi $(a+b)^n$ in Seriem expansi" and was found bound with one copy of Moivre's "Miscellanea Analytica," 1730. Pearson remarks:

"Many copies of this work have attached to them a *Supplementum* with separate pagination, ending in a table of 14 figure logarithms of factorials from 10! to 900! by differences of 10. But only a very few copies [P. tells of only the one] have a second supplement, also with separate pagination (pp. 1–7) and dated Nov. 12, 1733. This second supplement could only be added to copies sold three years after the issue of the original book, and this accounts for its rarity. Dr. Todhunter in writing his *History of the Theory of Probability* appears to have used the 1730 issue of Miscellanea Analytica, and so never came across this supplement."

Pearson here appears to make two slips: (*a*) in assuming that because this "second supplement" is bound at the end of a copy in the University College Library, it really was a second supplement; (*b*) in stating that this publication was not considered in Todhunter's "History." Curiously enough, in the latter part of his first article, after having indicated the results of the 'second supplement,' Pearson remarks: "The same matter is dealt with twenty-three years later in the edition of *The Doctrine of Chances*, pp. 243–250. Todhunter in his *History of the Theory of Probability*, Arts. 324 and 335, passed over the topic most superficially." Did Pearson overlook that a translation of his 'supplement' was thus dealt with in 1756 by Moivre and in 1865 by Todhunter? That this translation appeared also eighteen years before was manifestly not recognised.

Moreover, Moivre prefaces his translation with the statement (which Todhunter quotes): "I shall here translate a Paper of mine which was printed November 12, 1733, and was communicated to some Friends, but never yet made public, reserving to myself the right of enlarging my own thoughts as occasion shall require." Hence it is clear that the publication in question was not a 'second supplement' to Moivre's "Miscellanea Analytica," but was first 'made public,' and in English, in 1738. A facsimile of this pamphlet is to appear in an early number of *Isis*. It would be interesting to learn if any other copy of Moivre's original pamphlet is in existence.

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February 22.

I am glad that Prof. Archibald's letter enables me to return to the subject of De Moivre's claim to be the first discoverer of the normal curve of errors usually attributed to Laplace or Gauss.

I do not think from what Prof. Archibald has written that he can have seen what he thinks – and possibly may be – the unique copy of the "Approximatio ad Summam Terminorum Binomi." It is in the same type and has the same characteristic species of tailpiece as the "Miscellanea Analytica." It has the same unusual form of pagination as the first "Supplementum"; it is printed on the same paper and is of the same format as the first "Supplementum" and the "Miscellanea." About half the known copies of the "Miscellanea Analytica" have not the first supplement, and I think it quite probable that the "Approximatio" was only bound up with a few last copies of the "Miscellanea Analytica" issued after November 1733. At any rate, that is where I should seek for it first. I said in my paper that De Moivre treated of the same subject in his "Doctrine of Chances," 1756, because that was the edition I was working with. Prof. Archibald says that it was also treated of in the 1738 edition, and then speaks as if the matter in the 1738 "Doctrine of Chances" and again in the 1756 edition was a mere translation "except for minor changes." This is not correct; for the history of statistics most important additions were made in both the 1738 and the 1756 editions. The important principle of the 'activating deity' maintaining the stability of statistical ratios does not appear in the 1733 "Approximatio"; it first appears tentatively in the 1738 "Doctrine," where the seven lines of Corollary X are increased to nearly fifty lines, while in the 1756 "Doctrine" this corollary alone occupies some four pages or about 160 lines. Indeed the 6 pages of the "Approximatio" becomes $11\frac{1}{2}$ pages (of more lines) in the 1756 "Doctrine." As De Moivre appropriately observes, he has reserved to himself "the right of enlarging my own thoughts." That enlargement, developing Newton's idea of an omnipresent activating deity, who maintains mean statistical values, formed the foundation of statistical development through Derham, Süssmilch, Niewentyt, Price to Quetelet and Florence Nightingale. These may be mathematically 'minor' points, but they are vital for the history of statistics, and my reference to these additions in the penultimate paragraph of my paper might have shown Prof. Archibald that I was aware of the differences between the original "Approximatio" and the same dealt with in the "Doctrine of Chances." My error lay in not recognising that in the 1738 "Doctrine," the $6\frac{1}{2}$ pages had grown to a little over 8, to become $11\frac{1}{2}$ pages eighteen years later.

As to Dr. Todhunter, I have nothing whatever to retract in my judgement. In his Art. 335 he misses entirely the epoch-making character of the "Approximatio" as well as its enlargement in the "Doctrine." He does not say "Here is the original of Stirling's Theorem, here is the first appearance of the normal curve, here De Moivre anticipated Laplace as the latter anticipated Gauss. He does not even refer to the manner in which De Moivre expanded the Newtonian theology and directed statistics into the channel down which it flowed for nearly a century. Almost everywhere in his "History" Todhunter seizes a small bit of algebra out of a really important memoir and often speaks of it as a school exercise, whereas the memoir may have exerted by the principles involved a really wide influence on the development of the mathematical theory of statistics, and ultimately on statistical practice also.

Todhunter fails almost entirely to catch the drift of scientific evolution, or to treat that evolution in relation to the current thought of the day, which influences science as much as science influences general thought. The causes which led De Moivre to his “Approximatio” or Bayes to his theorem were more theological and sociological than purely mathematical, and until one recognizes that the post-Newtonian English mathematicians were more influenced by Newton’s theology than by his mathematics, the history of science in the eighteenth century – in particular that of the scientists who were members of the Royal Society – must remain obscure. KARL PEARSON.

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A Rare Pamphlet of Moivre and some of his Discoveries

As a result of two recent studies by KARL PEARSON[1] valuable information is at hand for forming a true estimate of the important position which ABRAHAM DE MOIVRE occupies in the history of mathematics. These studies are based on a consideration of : (a) part 4 of JEAN BERNOULLI's *Ars Conjectandi*[2], which contains what is properly called ≪ Bernoulli's Theorem ≫; and (b) a pamphlet of MOIVRE dated November 12, 1733.

The chief purpose of this paper is to summarize some of PEARSON's results, to remove misapprehension which one of his pamphlets may cause, and to present a facsimile of the MOIVRE pamphlet which is entitled : *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi.*

PEARSON opens his article as follows :

≪ It is usual to attribute the discovery of the Normal curve of errors to GAUSS. This is solely due to the fact that LAPLACE's *Théorie analytique des Probabilités* was published in 1812, and to this most writers have referred. But LAPLACE's *Mémoire sur les Probabilités* was published in the *Histoire de l Académie des Sciences* in 1778, and this memoir contains the normal curve function, and emphasizes the importance of tabulating the probability integral. Nay, we go further and say that (*Mémoires . . . présentés* T. IV. p. 6) when discussing BAYES' Theorem had also reached the exponential curve of errors as an approximation to the hypergeometrical series. All GAUSS' work falls in the 19th century. His *Theoria motus corporum coelestium* was published in 1809, his theory of least squares and his theory of combination of observations being of a still later date. There is, I think, not a doubt that LAPLACE's name ought to be associated with the normal curve and the probability integral before GAUSS'.

„ But in studying DE MOIVRE I have come across a work which long antedates both LAPLACE and GAUSS.

„ The matter is a very singular one historically. DE MOIVRE published in 1730 his *Miscellanea Analytica*, still a mine of hardly fully explored wealth. Many copies of this work have attached to them a *Supplementum* with separate pagination, ending in a table of 14 figure logarithms of factorials from 10! to 900! by differences of 10. But only a *very few* copies [PEARSON found only one!] have a second supplement, also with separate pagination (pp. 1–7) and dated Nov. 12, 1733. This second supplement could only be added to copies sold three years after the issue of the original book, and this accounts for its rarity. Dr. TODHUNTER in writing his *History of the Theory of Probability* appears to have used the 1730 issue of *Miscellanea Analytica*, and so never come across this supplement. "

In this last paragraph PEARSON makes statements which might readily mislead : firstly, in suggesting that because this ≪ second supplement ≫ is bound at the end of a copy of *Miscellanea Analytica* in the University College, London, it really was a supplement ; secondly, in asserting that this publication was not considered in TOD-

[1]K. Pearson, " Historical note on the origin of the normal curve of errors, " *Biometrika*, vol. 16, pp. 402–404, Dec., 1924; " JAMES BERNOULLI's Theorem, " *Biometrika*, vol. 17, pp. 201–210, Dec., 1925.

[2]Published at Basle in 1713, eight years after BERNOULLI's death, by his nephew NICOLAS BERNOULLI. Pearson shows that ≪ Pars Quarta ≫ of the *Ars Conjectandi* has not the importance which has often been attributed to it.

HUNTER's *History*. As a matter of fact this ≪ second supplement ≫ appeared in English, except for minor changes (see below), in both the second and third editions of MOIVRE's *Doctrine of Chances*[3]. TODHUNTER considers this in paragraphs 324 and 335 of his History[4]. Moreover MOIVRE prefaces his translation with the statement : I shall here translate a Paper of mine which was printed *November* 12, 1733, and communicated to some Friends, but never yet made public, reserving to myself the right of enlarging my own Thoughts ,[[5]] as occasion shall require. Hence it is clear that the publication in question was not a ≪ second supplement ≫ to MOIVRE's *Miscellanea Analytica* but was first ≪ made public ≫ in 1738. Another copy of Moivre's original pamphlet is in the Preussiche Staatsbibliothek, Berlin.

PEARSON gives priority to MOIVRE in :

1. Presenting the first treatment of the probability integral, and essentially of the normal curve;

2. Formulating and using the theorem, improperly called Stirling's theorem;

3. Enunciating the theorem that the measure of accuracy depends on the inverse square root of the size of the sample, so often called BERNOULLI's theorem[6] although it is entirely due to MOIVRE.

Moreover for this theorem MOIVRE appreciated the immense range of application. ≪ For him it was a theological problem, he was determining the frequency of irregularities from the Original Design of the Deity. Without grasping this side of the matter, it is impossible to understand the history of statistics from DE MOIVRE through DERHAM and SÜSSMILCH to QUETELET, culminating in the modern principle of the stability of

[3]Second edition, London, 1738, pp. 235–242; third edition, 1756, pp. 243–250 after translation (pp. 250–254, 334) is interesting new material.

[4]Curiously enough in the latter part of his first article, after having indicated the results of the ≪ second supplement ≫, PEARSON remarks : ≪The same matter is dealt with twenty-three years later in the edition of 1756 of the *Doctrine of Chances*, pp. 243–250. TODHUNTER in his *History of the Theory of Probability*, Arts, 324 and 335, passed over this topic most superficially.≫ Did Pearson overlook it was a translation of his ≪ supplement ≫ that was dealt with in 1756 by MOIVRE, and in 1865 by TODHUNTER? That this translation appeared also 18 years before was manifestly not recognized in the articles in question.

[5][In the 1738 translation there is practically no change from the original before corollary 4, where there is a slight alteration in the first clause ; corollary 5 is more than doubled in length by the addition of a third paragraph ; additions have been made near the first and last of corollary 6 and the odds " 792 ad 1 proxime " have been changed to "369 to 1 nearly "; by additions lemma 2 has been more than trebled in length; corollary 9 is slightly, but corollary 10 greatly, extended.]

[6]A correct statement of Bernoulli's theorem, given on page 236, of *Ars Conjectandi,* is as follows : Sit igitur numerus casuum fertilium ad numerum sterilium vel præcisè, vel proximè in ratione r/s, adeoque ad numerum omnium in ratione $r/(r+s)$ seu r/t, quam rationem terminent limites $(r+1)/t$ $(r-1)/t$. Ostendum est, tot posse capi experimenta, ut datis quotliber (puta c) vicibus, versimilius evadut, numerum fertilium observationum intra hos limites quàm extra casurum esse, h.e. numerum fertilium ad numerum omnium observationum rationem habiturum nec majorem quàm $(r+1)/t$, nec minorem quà $(r-1)/t$. [Therefore, let the number of fertile cases to the number of sterile cases be exactly or approximately in the ratio r to s, and hence the ratio of fertile cases to all the cases will be $r/(r+s)$ or r/t, which is within the limits $(r+1)/t$ and $(r-1)/t$. It must be shown that so many trials can be run such that it will be more probable than any given times (e.g., c times) that the number of fertile observations will fall within these limits rather than outside these limits – i.e., it will be c times more likely than not that the number of fertile observations to the number of all the observations will be in a ratio neither greater than $(r+1)/t$ nor less than $(r-1)/t$.]

statistical ratios. No one had any true inkling of the ideas of probable deviation of the statistical ratio before DE MOIVRE ≫.

Pearson concludes his second paper as follows : ≪ BERNOULLI saw the importance of a certain problem ; so did PTOLEMY, but it would be rather absurd to call KEPLER's or NEWTON's solution of planetary motion by PTOLEMY's name! Yet an error of like magnitude seems to me made when DE MOIVRE's method is discussed without reference to its author, under the heading of ≪ BERNOULLI's Theorem ≫. The contribution of the BERNOULLIS to mathematics is considerable, but they have been in more than one instance greatly exaggerated. The *Pars Quarta* of the *Ars Conjectandi* has not the importance which has been attributed to it. ≫

Now as to a second theorem for which MOIVRE deserves credit, namely the one wrongly associated with the name of STIRLING, MOIVRE finds the ratio of the maximum terms of a binomial to the term at a distance x from the maximum.

≪He supposes his power so large, that we may practically use

$$m! = \textit{const.} \times \sqrt{m}\, e^{-m} m^m .$$

He determines the constant which he calls B by the theorem that the hyperbolic logarithm of B is given by :

$$\log_e B = 1 - 1/12 + 1/360 - 1/1260 + 1/1680, \text{ etc.},$$

which gives $\log B = .399, 2235$ to the terms written down, or $B = 2.5074$.

Thus

$$m! = 2.5074 \times \sqrt{m}\, e^{-m} m^m . \gg$$

Then MOIVRE states : ≪ When I first began that inquiry, I contented myself to determine at large the Value of B, which was done by the addition of some Terms of the above-written Series; but as I perceiv'd that it converged but slowly, and seeing at the same time that what I had done answered my purpose tolerably well, I desisted from proceeding further, till my worthy and learned Friend Mr. JAMES STIRLING, who had applied himself to that inquiry, found that the quantity B did denote the Square-root of the Circumference of a Circle whose radius is unity ≫ PEARSON remarks very rightly : ≪ I consider that the fact that STIRLING showed that DE MOIVRE's arithmetical constant was $\sqrt{2\pi}$ does not entitle him to claim the theorem, and it is erroneous to term it STIRLING's Theorem [7]

[7] J. STIRLING, *Methodus Differentialie*, London 1730, p. 137 ; English edition by F. Holliday, London, 1749, p. 121. While STIRLING gave, in effect, the formula, a being the reciprocal of NAPIER's logarithm of 10 :

$$\operatorname{Log} x! = \frac{1}{2}\log 2\pi + (x+\frac{1}{2})\log(x+\frac{1}{2}) - (x+\frac{1}{2})a - \frac{a}{2.12.(x+\frac{1}{2})} + \frac{a}{8.360\,(x+\frac{1}{2})^3}$$

with the law for the continuation of the series, Moivre expressed the result, in effect, in the more convenient form (compare, C. TWEEDIE, *James Stirling*, Oxford, 1922, p. 119, 203–205) :

$$\log x! = \frac{1}{2}\log 2\pi + (x+\frac{1}{2})\log x - x + \frac{B_1}{1.2}\frac{1}{x} + \frac{B_3}{3.4}\frac{1}{x^3} + \dots$$

B_1 , B_3 denoting the BERNOULLI numbers. But in spite of the fact that it was MOIVRE, and not STIRLING, A facsimile copy of the original pamphlet follows[8].

(Brown University, Providence, R. I.) R. C. ARCHIBALD

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[The text of the Supplementum *can be found in A De Moivre,* The Doctrine of Chances *(2nd ed.), London: H Woodfall 1738, reprinted London: Cass 1967, or A De Moivre,* The Doctrine of Chances *(3rd ed.), London: A Millar 1756, reprinted New York, NY: Chelsea 1967 with a biographical article from* Scripta Mathematica ***2*** *(4) (1934), 316-333, by H M Walker. It is also reprinted in D E Smith,* A Source Book in Mathematics, *2 vols, New York, NY: McGraw-Hill 1929, reprinted New York, NY: Dover 1959, and in Appendix 5 of F N David,* Games, Gods and Gambling*, London: Griffin 1962.]*

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who gave the series

$$\frac{B_1}{1.2}\frac{1}{x} + \frac{B_3}{3.4}\frac{1}{x^3} + \dots$$

this series has been called STIRLING's series by Godefroy, *Théorie élémentaire des Séries*, Paris, 1903, pp. 224–228, and others. Compare *Encylopädie der mathematischen Wissenschaften*, vol. 1, part 2, 1900–1904, p. 931.

[8] I am indebted to the Librarian of University College, London, for his courtesy in allowing a photostatic copy to be made. Should any reader know of other copies of the original, he would confer a favor on the writer by informing him where they may be found.