# CUBIC AND QUARTIC EQUATIONS

From the point of view of medieval mathematicians, there are actually 13 different types of cubic equations rather than just one. Basically this is because they not merely did not admit imaginary or complex numbers, but only considered positive real numbers, so also did not admit negative numbers or zero. Thus to them $x^3 + x = 6$ was a different type of equation from $x^3 = x + 6$. Now we can write a general cubic equation

$$ay^3 + by^2 + cy + d = 0$$

(in which $a \neq 0$ or the equation is not genuinely cubic) in the form

$$y^3 + by^2 + cy + d = 0$$

after dividing by a constant. The case where $d = 0$ leads to an inadmissible root $y = 0$ and anyway is easily soluble, so we get different cases as

$$b \begin{array}{c} > \\ = \\ < \end{array} 0, \qquad c \begin{array}{c} > \\ = \\ < \end{array} 0, \qquad d \begin{array}{c} > \\ = \\ < \end{array} 0$$

leading to 18 cases. However, the cases

$$\begin{array}{ll} b = c = 0, & d > 0 \\ b = c = 0, & d < 0 \end{array}$$

are not taken seriously as cubic equations (and indeed the first has no real positive solution), while evidently the cases

$$\begin{array}{lll} b > 0, & c > 0, & d > 0 \\ b = 0, & c > 0, & d > 0 \\ b > 0, & c = 0, & d > 0 \end{array}$$

have no real positive solution. This leaves 13 cases to be considered.

The case of the cosa and the cube, in modern notation the case $y^3 + cy = d$ where $c$ and $d$ are positive, was solved by Scipione del Ferro (1465–1626) early in the sixteenth century. He taught his method to his pupil Antonio Maria Fior (Florido) (dates unknown), who had a contest with Nicolò Tartaglia (1500–1557) which resulted in the latter's discovery of the method for solving this particular type. Girolamo Cardano (Jerome Cardan) (1501–1576) persuaded Tartaglia to tell him the solution, first in a cryptic verse and then with a full explanation, after swearing he would keep the solution secret. But, after he had found the solution in the posthumous papers of del Ferro, Cardan felt free to publish, which he did in his *Ars Magna* (1545). However, Cardan went further than his predecessors because he considered all 13 forms successively.

We can express Cardan's approach in modern terms as follows. We first define $x = y + g$, so that the equation becomes

$$(x - g)^3 + b(x - g)^2 + c(x - g) + d = 0.$$

If we take $g = b/3$, the coefficient of $x^2$ vanishes so that the equation can be written in the form

$$x^3 + px + q = 0.$$

Now write $x = h + k$, so that

$$h^3 + k^3 + (p + 3hk)(h + k) + q = 0.$$

Clearly $h$ and $k$ will satisfy this equation if

$$\begin{aligned} h^3 + k^3 &= -q \\ hk &= -p/3. \end{aligned}$$

Setting $u = h^3$ and $v = k^3$ these equations amount to

$$\begin{aligned} u + v &= -q \\ uv &= -p^3/27 \end{aligned}$$

and solutions of these equations can be found by considering the quadratic

$$\xi^2 + q\xi - p^3/27 = 0$$

so that we can take $u$ and $v$ as

$$-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

As $u$ and $v$ enter into the problem symmetrically, we may as well take $u$ as the expression with the $+$ sign and $v$ as the expression with the $-$ sign. We can then take

$$\begin{aligned} h &= \sqrt[3]{u} \\ k &= -p/3h. \end{aligned}$$

Finally we can find $x$ as $h + k$, that is as

$$x = \sqrt[3]{u} + \sqrt[3]{v}.$$

If we suppose the roots are $x_1$, $x_2$ and $x_3$, then the equation must be equivalent to

$$(x - x_1)(x - x_2)(x - x_3) = 0$$

so that

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2x_3 + x_3x_1 + x_1x_2 &= p \\ x_1x_2x_3 &= -q \end{aligned}$$

and hence it can shown that

$$[(x_1 - x_2)(x_3 - x_1)(x_2 - x_3)]^2 = -4p^3 - 27q^2$$

(cf. Chapter 5, §6 of the book by Birkhoff and MacLane quoted below). The latter quantity is often referred to as the *discriminant* and denoted $D$. We see that the above expressions for $u$ an $v$ can be written as

$$-q/2 \pm \sqrt{-D/108}$$

so that

$$x = \sqrt[3]{u} + \sqrt[3]{v} = \sqrt[3]{-q/2 + \sqrt{-D/108}} + \sqrt[3]{-q/2 - \sqrt{-D/108}}.$$

This is the celebrated formula of Cardan.

There is a difficulty in this solution which was first observed by Cardan, although he did not see the way round it. When the roots of the cubic are all real and distinct, then $D$ is real and positive, so that the above roots for $h$ and $k$ are complex. This means that the real roots can be expressed in terms of the cube roots of complex numbers. However, these real roots cannot be obtained by algebraic means, that is, by radicals. This case was called *irreducible* by Tartaglia. (For example $-6$ is a root of $x^3 = 63x + 162$ which Cardan's expression derives as $\sqrt[3]{81 + 30\sqrt{-3}} + \sqrt[3]{81 - 30\sqrt{-3}} = (-3 + 2\sqrt{-3}) + (-3 - 2\sqrt{-3})$.) In fact the *Ars Magna* included several complex roots of quadratics, but Cardan says of them, "So progresses arithmetic sublety the end of which, as is said, is as refined as it is useless" (Chapter XXXVII). Cardan also discussed the number of roots to be expected in a cubic and began the study of symmetric functions.

Since every number has three distinct cube roots, we have evidently obtained several values of $x$. This is as it should be, for a cubic equation usually has three distinct roots. But at first sight it appears that there are nine, or even eighteen, possibilities in the formula, since alternative solutions exist for square roots and also for cube roots. As for square roots, a glance shows that the signs are fixed—one must be positive and one negative. Accordingly we consider the cube roots.

Let the distinct roots of $x^3 - 1 = 0$ be $1, \omega, \omega^2$, so that

$$1 + \omega + \omega^2 = 0, \qquad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Then $\omega^3 = 1$ and we can replace $h$ by $\omega^i h$ and $k$ by $\omega^j k$ where $i, j = 1, 2, 3$. Now $h$ and $k$ were constructed to satisfy

$$\begin{aligned} h^3 + k^3 &= -q \\ hk &= -p/3 \end{aligned}$$

but while

$$(\omega^i h)^3 + (\omega^j k)^3 = -q$$

for any $i, j$, the equation

$$(\omega^i h)(\omega^j k) = -p/3$$

holds only if $i + j \equiv 0 \mod 3$, reducing us to three possibilities.

Given a quartic (or biquadratic) equation

$$ay^4 + by^3 + cy^2 + dy + e = 0$$

(where $e$ is not necessarily the base of natural logarithms) in which $a \neq 0$, we can divide through by a constant, so that we can act as if $a = 1$. We then define $x = y + g$, so that the equation becomes

$$(x-g)^4 + b(x-g)^3 + c(x-g)^2 + d(x-g) + e = 0.$$

If we take $g = b/4$, the coefficient of $x^3$ vanishes so the equation can be written in the form

$$x^4 + px^2 + qx + r = 0.$$

Now the left-hand side of

$$x^4 + px^2 = -qx - r$$

contains two of the terms of the square of $x^2 + p$. Complete the square by adding $px^2 + p^2$ to each side to get

$$(x^2+p)^2 = x^4 + 2px^2 + p^2 = px^2 + p^2 - qx - r.$$

We now introduce another unknown for the purpose of converting the left-hand side of the this equation into $(x^2 + p + z)^2$. This is done by adding $2(x^2 + p)z + z^2$ to each side, and leads to

$$\begin{aligned}(x^2+p+z)^2 &= px^2 + p^2 - qx - r + 2(x^2+p)z + z^2 \\ &= (p+2z)x^2 - qx + (p^2 - r + 2pz + z^2).\end{aligned}$$

The problem now reduces to finding a value of $z$ that makes the right-hand side, a quadratic in $x$, a perfect square. This will be the case when the discriminant of the quadratic is zero; that is, when

$$q^2 = 4(p+2z)(p^2 - r + 2pz + z^2),$$

which requires solving a cubic in $z$, namely

$$8z^3 + 20pz^2 + (16p^2 - 8r)z + (4p^3 - 4pr - q^2) = 0.$$

The last equation is known as the *resolvent cubic* of the given quartic equation, and it can be solved as described above. There are in general three solutions of the resolvent cubic, and $x$ can be determined from any one of them by extracting square roots. Once a value of $x$ is known, the solution of the original quartic is readily deduced.

An expression for the quartic discriminant is given by Turnbull in equation (12) on p. 123 of the book quoted below.

The solution of the quartic was first given by Ludovico Ferrari (1522–1565).

## *References*

G Birkhoff and S. MacLane, *A Survey of Modern Algebra*, New York, NY: Macmillan 1941, 1953 and 1965, Chapter V, §§5-6 and Chapter XV, §7.

D M Burton, *The History of Mathematics: An Introduction*, Boston, MA, etc: Allyn and Bacon 1985, Chapter 7.

G Cardano, *Ars Magna or The Rules of Algebra*, translated and edited by T.R. Witmer, Cambridge, MA: MIT Press 1968 and New York, NY: Dover 1993.

J Fauvel and J Gray, *The History of Mathematics: A Reader*, Macmillan: Basingstoke and London 1987, Chapter 8, §8A.

S Neumark, *Solution of Cubic and Quartic Equations*, Oxford, etc.: Pergamon 1965.

H W Turnbull, *Theory of Equations*, Edinburgh and London: Oliver and Boyd 1939, 1944, 1946, 1947 and 1952, Chapters IX and X.

B L van der Waerden, *Modern Algebra*, New York, NY: Ungar 1949 and 1953, Volume I, Chapter VII, §58.

P.M.L.