MÉMOIRE
SUR LES
APPROXIMATIONS DES FORMULES
QUI SONT FONCTIONS DE TRÈS GRANDS NOMBRES
(SUITE)

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This Memoir being a sequel to that which has appeared under the same object in the preceding Volume, I will conserve the order of the articles and of the sections. I have given, in the first article, a general method to reduce to highly convergent series the differential functions which contain some factors raised to great powers. In the second article, I have restored to this kind of integrals all the functions given by some equations linear in the ordinary or partial differences, finite and infinitely small; and I am thus arrived, in the third article, to determine the approximate values of many formulas which are encountered frequently in Analysis, but of which the application becomes very painful when the numbers of which they are functions are large. There remains to me presently to show the usage of this analysis in the theory of chances.

ARTICLE IV.
Application of the preceding analysis to the theory of chances.

XXXII.

All events, even those which by their smallness and their irregularity seem to not depend upon the general system of nature, are a series as necessary as the revolutions of the Sun. We attribute them to chance, because we are ignorant of the causes which produce them and the laws which link them to the great phenomena of the universe; thus the apparition and the movement of comets, which we know today depend on the same law which restores the seasons, was regarded yesterday as the effect of chance by those who arranged these stars among the meteors. The word *chance* expresses thus only our ignorance of the causes of the phenomena which we see to happen and to succeed themselves without any apparent order.

The probability is relative in part to this ignorance, in part to our knowledge. We know, for example, that out of three, or a greater number of events, one alone must exist; but nothing brings to belief that one of them will arrive rather than the others. In this state of indecision, it is impossible to pronounce with certitude on their existence. It seems to us however probable that one of these events, taken at will, will not exist, because we see many cases equally possible which exclude its existence, while one alone favors it.

The theory of chances consists therefore to deduce all the events which can take place relatively to an object, into a certain number of equally possible cases, that is such as we are equally undecided on their existence, and to determine the number of the cases favorable to the event of which we seek the probability. The ratio of this number to the one of all the possible cases is the measure of this probability.

All our judgments on the things which are only probables are founded on the parallel ratio: the difference of the facts which each man has on them and the errors which we commit in evaluating this ratio give birth to that crowd of opinions which we see reign on the same objects; the combinations of this type are so delicate and the illusions so frequent, that a great attention is often necessary to escape the error.

The theory of chances offers a great number of examples, in which the results of Analysis are entirely contrary to those which present themselves at first glance, that which proves how often it is useful to apply the calculus to the important objects of civil life; and, when even the possibility of these applications would oblige to make some hypotheses which would be only approximate, the precision of the analysis renders always the results of it preferable to the vague reasonings which we employ often to treat these objects.

The preceding notion of the probability gives a quite simple solution to a question agitating to some philosophers, and which consists in knowing if the past events influence on the probabilities of the future events. We suppose that in the game of *croix et pile* we have brought forth *croix* more often than *pile*; by that alone we will be brought to believe that, either in the constitution of the coin, or in the manner of casting it, there exists a constant cause which favors the first of these events; the past trials have then one influence on the probability of the future trials; but, if we are assured that the two faces of the coin are perfectly similar, and if moreover the circumstances of its projection are at each trial varied, in a way that we are restored without ceasing to the state of an absolute indecision on that which must happen, the past can have no influence on the probability of the future, and it will be evidently absurd to take account of it.

When the possibility of the simple events is known, the probability of the composite events can often be determined by the sole theory of combinations; but the most general method in order to attain it consists in observing the law of the variations which it sustains by the multiplication of the simple events, and to make it depend on one equation in the ordinary or partial finite differences: the integral of this equation will give the analytic expression of the sought probability. If the event is so composite that the use of this expression becomes impossible, because of the great number of its terms and of its factors, we will have its approximate value by the method exhibited in the preceding articles. We will see an example at the end of this Memoir.

In a great number of cases, and these are the most interesting in the analysis of chances, the possibilities of the simple events are unknown, and we are reduced to seek in the past events some indices which can guide us in our conjectures on the future. But in what manner do these events unfold to us, in expanding themselves, their respective possibility? According to what laws do they influence on the probability of future events? These are some difficult questions, of which the solution requires some very delicate metaphysical considerations and a sensitive analysis. The difficulty of solving them makes itself felt principally when the question is to ascertain some small differences through the observations, because then a considerable number of observed events can indicate only these differences with a very small probability; and, if we use these events in very great number, we are lead to some formulas of which it is impossible to make use. It is therefore indispensable then to have a simple means to obtain the law according to which the probability of a result indicated by the observations increases with them, and the number to which the observed events must be raised in order that, this result acquiring a great probability, we are justified to research the causes which produce it. I have given moreover the principles and the method necessary for this object, and this method has the advantage of being as much more precise as the observed events are in greater number: the analysis exhibited in the preceding articles having lead me to generalize it and to simplify it, I am going to present it here in a new day, by giving some very convenient formulas in order to determine, after the observation of results composed of a great number of simple events, the possibilities of these events, the differences that the time, the climate, or other causes can produce in it, and the probability of future events.

In order to clarify this method by an example, I will apply it to some problems on the births: it is an important object in the natural history of man, and the observation offers in this regard some remarkable varieties relatively to the difference of the sexes and of the climates; but they are so small in themselves that they can become sensible only by a great number of births. By comparing those which have been observed in the great cities, I find that from the north to the middle of Europe they indicate a greater possibility in the births of boys than in those of girls, with a probability so very near to certitude that there exists in natural philosophy no result better established by the observations. This superiority in the possibility of the births of boys is therefore a general law of nature, at least in the part of the globe that we inhabit; and, if we consider that it subsists despite the great varieties of climates and of productions, which take place from Naples to Petersburg, it will appear probable that this law extends to the whole Earth.

An equally interesting result and which the observations indicate with great probability is that the possibility of the births of boys, relatively to that of the births of girls, is not everywhere the same. It is here especially that it matters to have an easy method to compare a very great number of births and to determine the probability which results from it that the observed differences are not due to chance: these differences are so very small that often many millions of births are necessary to establish that they are the result of always active causes and that we must distinguish them from those small varieties which chance alone brings forth in the succession of the equally possible events. I give, in order to obtain this probability, some very simple formulas, by means of which we can immediately judge its magnitude: these formulas, applied to the births observed at London and at Paris, give a probability of more than four hundred thousand against one that the possibility of the births of boys compared to that of the births of girls is greater in the first of these two cities than in the second; whence it follows that there exists very probably in London a cause greater than in Paris which renders the births of the boys superior to those of girls. The births observed in the realm of Naples seems to indicate similarly in this realm a greater possibility than in Paris in the births of boys; but, although the sum of the observed births in these two places is elevated to more than two millions, this result is hardly indicated with a probability of one hundred to one. Thus, in order to pronounce irrevocably on this object, it is necessary to await a greater number of births.

XXXIII.

Whatever be the manner in which two events are linked the one to the other, it is clear that the probability of their sum is equal to the probability of the first, multiplied by the probability that, the one taking place, the second must similarly exist; we will have therefore this last probability in determining *a priori* the probability of the sum of two events and by dividing it by the probability of the first event determined *a priori*.

In order to express analytically this result, we name E and e the two events; $\mathrm{E}+e$ their sum; V the probability of E; v that of $\mathrm{E}+e$; and p the probability of e, by supposing that E exists. We will have, this put,

$$p = \frac{v}{\mathrm{V}}.$$

This quite simple equation is the basis of the following researches, and all the theory of the probability of causes and of the future events, taken from the past events, result from it with a great ease. Let us see first how it gives the respective probabilities of the different causes to which we can attribute an observed event.

XXXIV.

Let E be this event and we suppose that it can be attributed to the n causes e, $e^{(1)}$, $e^{(2)}, \dots, e^{(n-1)}$; if we name $p^{(r)}$ the probability of the cause $e^{(r)}$, taken from the event E, V the probability of E and v that of $\mathrm{E}+e^{(r)}$, we will have, by the preceding section,

$$p^{(r)} = \frac{v}{\mathrm{V}}.$$

It is necessary now to determine v and V; for this we will observe that the probability *a priori* of the existence of the cause $e^{(r)}$ is $\frac{1}{n}$; by naming therefore a, $a^{(1)}, a^{(2)}, \dots, a^{(n-1)}$ the respective probabilities that, the causes e, $e^{(1)}$, $e^{(2)}, \dots$ being supposed to exist, the event E will take place, $\frac{a^{(r)}}{n}$ will be the probability of $\mathrm{E}+e^{(r)}$ determined *a priori*: it is the quantity which we have named v.

The sum of all these probabilities relative to each of n causes will be evidently the probability of E, since this event can arrive only by one of these causes; we will have therefore

$$\mathrm{V} = \frac{1}{n}(a + a^{(1)} + \cdots + a^{(n-1)}),$$

hence

$$p^{(r)} = \frac{a^{(r)}}{a + a^{(1)} + a^{(2)} + \cdots + a^{(n-1)}},$$

that is that we will have the probability of one cause, taken from the event, by dividing the probability of the event, taken from that cause, by the sum of all the similar probabilities.

We suppose, for example, that an urn contains three balls which can be only white or black; that after having drawn from it a ball we remit it into the urn in order to proceed to a new drawing and that after m drawings we have brought forth only white balls: it is clear that we can make *a priori* only four hypotheses, because the balls will be entirely white or entirely black, or two will be white and one black, or two will be black and one white. If we consider these hypotheses as so many different causes e, $e^{(1)}$, $e^{(2)}, e^{(3)}$ of the observed event, the respective probabilities of this event, taken from these causes, will be 1, $\left(\frac{2}{3}\right)^m$, $\left(\frac{1}{3}\right)^m, 0$; these are the quantities which we have named a, $a^{(1)}$, $a^{(2)}$, $a^{(3)}$. The respective probabilities of these hypotheses, taken from the event, will be therefore, by the preceding formula,

$$\frac{3^m}{3^m + 2^m + 1}, \quad \frac{2^m}{3^m + 2^m + 1}, \quad \frac{1}{3^m + 2^m + 1}, \quad 0.$$

We see, besides, that it is useless to have regard to the hypotheses which exclude the event, because, the probability of the event resulting from these hypotheses being null, their omission changes not at all the value of $p^{(r)}$.

XXXV.

The possibility of most of the simple events is unknown and, considered *a priori*, it seems to us equally susceptible of all the values from zero to unity; but, if we have observed a result composed of many of these events, the manner in which they enter it renders some of these values more probable than the others.

Thus, in measure as the observed result is composed by the expansion of simple events, their true possibility is made more and more known, and it becomes more and more probable that it falls within some limits which are narrowed without ceasing and ending by coinciding when the number of simple events is infinite. In order to determine the laws according to which this possibility is discovered, we will name it x. The known theory of chances will give the probability of the observed result in a function of x; let y be this function. If we regard the different values of x as so many causes of the observed result, the probability of x will be, by No. XXXIV, equal to a fraction of which the numerator is y and of which the denominator is the sum of all the values of y. By multiplying therefore the two terms of this fraction by dx, this probability will be $\frac{y\,dx}{\int y\,dx}$, the integral of the denominator being taken from $x = 0$ to $x = 1$.

The probability that x is contained between the two limits $x = \theta$ and $x = \theta'$ is, consequently, equal to $\frac{\int y\,dx}{\int y\,dx}$, the integral of the numerator being taken from $x = \theta$ to $x = \theta'$ and that of the denominator being taken from $x = 0$ to $x = 1$.

The most probable value x is that which renders y a maximum; we will designate it by a: the least probable values are those which render y null. In nearly all the cases, this happens at the two limits $x = 0$ to $x = 1$. Thus we will suppose y null at these limits, and then each value of y will be a corresponding value which will be equal to it at the other side of the maximum.

If the values of x, considered independently of the observed result, are not all equally possible, but that their probability is expressed by a function z of x, it will suffice to change, in the preceding formulas, y into yz, that which returns to supposing all the values of x equally possible and to considering the observed result as being formed of two independent results, of which the probabilities are y and z. We can therefore restore in this manner all the cases to those where we suppose an equal possibility to the different values of x and, by this reason, we will adopt this hypothesis in the following researches.

XXXVI.

We will consider a result composed of a very great number of simple events and suppose that, after the observation of this result, we wish to have the probability that the possibility x of these events not surpass any quantity θ less than a; this probability is, by the preceding section, equal to the fraction $\frac{\int y\,dx}{\int y\,dx}$, the integral of the numerator being taken from $x = 0$ to $x = \theta$ and that of the denominator being taken from $x = 0$ to $x = 1$. We will have these integrals in a highly convergent series by the formulas of No. VI. If we make first $-\frac{y\,dx}{dy} = v$, and if we designate by U and J that which v and y become when we change x into θ, formula (*a*) of this section will give, for the expression in series of the integral $\int y\,dx$ taken from $x = 0$ to $x = \theta$,

$$\int y\,dx = -\mathrm{UJ}\left[1 + \frac{d\mathrm{U}}{d\theta} + \frac{d(\mathrm{U}\,d\mathrm{U})}{d\theta^2} + \cdots\right].$$

If we name next Y the maximum of y or that which this function becomes when we change x into a, if we make

$$\frac{x-a}{\sqrt{\log \mathrm{Y} - \log y}} = u,$$

these logarithms being hyperbolic, and if we designate by U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}, \dots$ that which u, $\frac{du^2}{dx}$, $\frac{du^3}{dx^2}, \dots$ become when we change x into a, formula (*d*) of the same section will give for the expression in series of the integral $\int y\,dx$, taken from $x = 0$ to $x = 1$,

$$\int y\,dx = \mathrm{Y}\sqrt{\pi}\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2}\frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right),$$

π being the ratio of the semi-circumference to the radius. The probability that x is equal or less than θ will be therefore

$$(a') \qquad \frac{-\mathrm{UJ}\left[1 + \frac{d\mathrm{U}}{d\theta} + \frac{d(\mathrm{U}\,d\mathrm{U})}{d\theta^2} + \cdots\right]}{\mathrm{Y}\sqrt{\pi}\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2}\frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right)}.$$

The numerator of this series forms a divergent series of θ is very near to a; in this case, we will have the integral $\int y\,dx$ from $x = 0$ to $x = \theta$ by formula (*c*) of No. VI, and we will find for the expression in series of this integral

$$\int y\,dx = \mathrm{Y}\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right)\int dt\,e^{-t^2}$$
$$-\frac{\mathrm{Y}e^{-\mathrm{T}^2}}{2}\left(\frac{dU^2}{dx} + \mathrm{T}\frac{d^2U^3}{1.2\,dx^2} + \cdots\right),$$

the integral relative to t being taken from $t = \mathrm{T}$ to $t = \infty$, T being given by the equation

$$\mathrm{T}^2 = \log \mathrm{Y} - \log \mathrm{J},$$

in which the logarithms are hyperbolic, and e being the number of which the hyperbolic logarithm is unity. The probability that x is equal or less than θ will be therefore given by this formula

$$(b') \qquad \frac{\int dt\, e^{-t^2}}{\sqrt{\pi}} - \frac{e^{-\mathrm{T}^2}\left(\frac{dU^2}{dx} + \mathrm{T}\frac{d^2U^3}{1.2\,dx^2} + \cdots\right)}{2\sqrt{\pi}\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \cdots\right)}.$$

We can, in every case, determine by means of formulas (a') and (b') the probability that x is equal or less than θ, θ being smaller than a.

If θ surpasses a, we will make $1-\theta=\theta'$, $1-x=x'$ and, by naming y' that which y becomes, we will seek the probability that x' is equal or less than θ' by the formula $\frac{\int y'\,dx'}{\int y'\,dx'}$, in which the integral of the numerator is taken from $x'=0$ to $x'=\theta'$, that of the denominator being taken from $x'=0$ to $x'=1$. Formulas (a') and (b') will give this probability, by changing $y,\ u,\ v,\ \theta$ into y', u', v', θ'; by subtracting it from unity next, we will have the probability that x is equal or less than θ.

The integral $\int dt\, e^{-t^2}$ is encountered frequently in this analysis, and, for this reason, it will be very useful to form a Table of its values, from $t=\infty$ to $t=0$. When this integral is taken from $t=\mathrm{T}$ to $t=\infty$, T being equal or greater than 3, we can make use of the formula

$$(c') \qquad \int dt\, e^{-t^2} = \frac{e^{-\mathrm{T}^2}}{2\mathrm{T}}\left(1 - \frac{1}{2\mathrm{T}^2} + \frac{1.3}{4\mathrm{T}^4} - \frac{1.3.5}{8\mathrm{T}^6} + \cdots\right),$$

which will give a value alternately greater or lesser than the true.

XXXVII.

We will determine the probability that the value of x is contained between the two limits $a-\theta$ and $a+\theta'$, which embraces the value of a corresponding to the maximum of y. This probability is equal to $\frac{\int y\,dx}{\int y\,dx}$, the integral of the numerator being taken from $x=a-\theta$ to $x=a+\theta'$, and that of the denominator being taken from $x=0$ to $x=1$.

We suppose θ and θ' very small and such that the two values of y, corresponding to $x=a-\theta$ and to $x=a+\theta'$, are equal to one same quantity which we will designate by J; formula (c) of No. VI will give, very nearly

$$\int y\,dx = \mathrm{Y}U\int dt\,e^{-t^2},$$

the integral relative to x being taken from $x = a - \theta$ to $x = a + \theta'$, and the integral relative to t being taken from $t = -\sqrt{\log \mathrm{Y} - \log \mathrm{J}}$ to $t = \sqrt{\log \mathrm{Y} - \log \mathrm{J}}$; the sought probability will be therefore equal to $\frac{\int dt\,e^{-t^2}}{\sqrt{\pi}}$.

y being supposed to have for factors some very elevated powers, the exponents of these powers become coefficients in its logarithm, so that, if we designate by α a very small fraction, $\log y$ will be of order $\frac{1}{\alpha}$, and $\sqrt{\log \mathrm{Y} - \log \mathrm{J}}$ will be of order $\frac{1}{\alpha^{\frac{1}{2}}}$, at least when J is very little different from Y.

We suppose that it differs from it rather little in order that $\sqrt{\log \mathrm{Y} - \log \mathrm{J}}$ is equal to $\frac{1}{\alpha^{\frac{\lambda}{2}}}$, λ being positive and less than unity; if we reduce $\log \mathrm{J}$ into a series ordered with respect to the powers of θ, the function $\sqrt{\log \mathrm{Y} - \log \mathrm{J}}$ will become of this form $\frac{\theta \mathrm{Q}}{\alpha^{\frac{1}{2}}}$; thus, in order that it be of order $\frac{1}{\alpha^{\frac{\lambda}{2}}}$, it is necessary that θ be quite small of order $\alpha^{\frac{1-\lambda}{2}}$; we will prove the same thing relatively to θ'. The interval $\theta + \theta'$ contained between the two limits $a - \theta$ and $a + \theta'$ will be therefore of order $\alpha^{\frac{1-\lambda}{2}}$; it will be consequently as much less as the events will be more multiplied, so that it will become null if their number is infinite, and, in this case, the two limits will confound themselves with the value of a which corresponds to the maximum of y.

In order to have the probability that the value of x is contained within these limits, it is necessary to determine the integral $\int dt\,e^{-t^2}$ from $t = -\frac{1}{\alpha^{\frac{\lambda}{2}}}$ to $t = \frac{1}{\alpha^{\frac{\lambda}{2}}}$. This integral is evidently the double of the integral $\int dt\,e^{-t^2}$ taken from $t = 0$ to $t = \infty$, less the double of that same integral taken from $t = \frac{1}{\alpha^{\frac{\lambda}{2}}}$ to $t = \infty$; now we have, by No. IV,

$$\int dt\,e^{-t^2} = \frac{1}{2}\sqrt{\pi},$$

the integral being taken from $t = 0$ to $t = \infty$; we have moreover, by formula (c') of the preceding section,

$$\int dt\, e^{-t^2} = \frac{1}{2}\alpha^{\frac{\lambda}{2}} e^{-\frac{1}{\alpha^\lambda}} \left(1 - \frac{\alpha^\lambda}{2} + \frac{3\alpha^{2\lambda}}{4} - \cdots\right);$$

the integral $\int dt\, e^{-t^2}$, taken from $t = -\frac{1}{\alpha^{\frac{\lambda}{2}}}$ to $t = \frac{1}{\alpha^{\frac{\lambda}{2}}}$, will be therefore

$$\sqrt{\pi} - \alpha^{\frac{\lambda}{2}} e^{-\frac{1}{\alpha^\lambda}} + \cdots$$

By dividing it by $\sqrt{\pi}$, we will have the probability that x is contained between the limits $a - \theta$ and $a + \theta'$; the expression of this probability will be, consequently,

$$(d') \qquad 1 - \frac{\alpha^{\frac{\lambda}{2}}}{\sqrt{\pi}} e^{-\frac{1}{\alpha^\lambda}} + \cdots$$

When $\frac{1}{\alpha}$ is a large number, this formula converges rapidly to unity, principally because of the factor $e^{-\frac{1}{\alpha^\lambda}}$, which becomes very small when α is a very small fraction; thence results this theorem:

The probability that the possibility of the simple events is contained between some limits which are contracted more and more approaches without ceasing to unity, in a manner that, under the supposition of an infinite number of simple events, these two limits coming to join themselves, and the probability is confounding with certitude, the true possibility of the simple events is exactly equal to that which renders the observed result the most probable.

We see thus how the events, by multiplying themselves, discover for us their respective possibility; but we must observe that there is in this analysis two approximations, of which the one is relative to the limits which contain the value of x and which are contracted more and more, and of which the other is relative to the probability that x is found between these limits, a probability which approaches without ceasing to unity or to certitude. It is in it that these approximations differ from ordinary approximations, in which we are always assured that the result is contained within the limits which we assign them.

It matters principally, in these researches, to be able to judge immediately if a result is indicated by the observations with a great possibility, because it suffices often to be assured that it is very probable, without that there be a need to know with much precision the value of the probability; by supposing therefore that the question is to determine if it is very probable that the possibility of a simple event is contained within some given limits, we can easily arrive to the following formula.

We have, by that which precedes,

$$\log \mathrm{Y} - \log \mathrm{J} = \frac{1}{\alpha^\lambda}.$$

Moreover, if we suppose θ very small, we have

$$\log \mathrm{J} = \log \mathrm{Y} + \theta \frac{d \log \mathrm{Y}}{dx} + \frac{\theta^2}{1.2} \frac{d^2 \log \mathrm{Y}}{dx^2} + \cdots;$$

but the condition of the maximum gives

$$\frac{d \log \mathrm{Y}}{dx} = 0, \qquad \frac{d^2 \log \mathrm{Y}}{dx^2} = \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2}:$$

we will have therefore

$$-\theta^2 \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2} = \frac{1}{\alpha^\lambda};$$

thus the probability that the possibility x of the simple event is contained between the limits $a - \theta$ and $a + \theta$ will be, by formula (d'),

$$1 - \frac{1}{\theta \sqrt{-\pi \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2}}} e^{\theta^2 \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2}} + \cdots;$$

whence we see that this probability will be very great if $-\theta^2 \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2}$ is a very small number, such as 11 or 12, that which gives a very simple way to judge the magnitude of this probability.

XXXVIII.

The possibility of the simple events can not be the same in different epochs or in some different countries: the climate, the productions and a thousand other physical and moral causes can produce the differences that a great number of observations render sensible; but, as the single combinations of chance suffice to introduce the slight differences in the result of the observations, we see that a very great number are necessary in order to be assured that the observed differences, when they are very small, are due to some always acting causes. This problem, one of the most important in the theory of chances, require a delicate analysis; here is a quite simple solution of it.

We suppose that we observe, in two different places, two results composed of a very great number of simple events of the same kind. Let

x be the possibility of the simple event in the first place;
y be the function of x which expresses the probability of the observed result in that place;
a be the value of x which corresponds to the maximum of y.

Let similarly

x' be the possibility of the simple event in the second place;
y' be the function of x' which expresses the probability of the observed result in that place;
a' be the value of x' which corresponds to the maximum of y';

a and a' are the possibilities of the simple events which render the observed results the most probable, and these quantities will be, by the preceding section, the true possibilities of the simple events, if the observed results were composed of an infinite number of these events. We suppose a' very little different from a, and that it is a little greater; finally we name P the probability that the possibility of the simple event is greater in the first than in the second. This put, we will have, by some considerations analogous to those of No. XXXV,

$$\text{P} = \frac{\int\int yy'\,dx\,dx'}{\int\int yy'\,dx\,dx'},$$

the integrals of the numerator being taken from $x' = 0$ to $x' = x$, and from $x = 0$ to $x = 1$; those of the denominator being taken from $x' = 0$ to $x' = 1$, and from $x = 0$ to $x = 1$.

In order to have these integrals, we will suppose $x' = ux$, and we will name z that which xyy' becomes when we will have

$$\text{P} = \frac{\int\int z\,dx\,du}{\int\int z\,dx\,du},$$

the integrals of the numerator being taken from $u = 0$ to $u = 1$, and from $x = 0$ to $x = 1$; those of the denominator being taken from $u = 0$ to $u = \frac{1}{x}$, and from $x = 0$ to $x = 1$. We determine first the integrals of the numerator.

For this, we will observe that, y being null at the two limits $x = 0$ and $x = 1$, z is similarly null at these two limits; let therefore Z be that which this function becomes when we substitute for x its value in u, given by the equation $0 = \frac{\partial z}{\partial x}$; we will have very nearly, by No. VI,

$$\int z\,dx = \frac{\sqrt{2\pi}\mathrm{Z}}{\sqrt{-\frac{d^2\mathrm{Z}}{\mathrm{Z}dx^2}}},$$

hence

$$\int\int z\,du\,dx = \sqrt{2\pi}\int \frac{\mathrm{Z}\,du}{\sqrt{-\frac{d^2\mathrm{Z}}{\mathrm{Z}dx^2}}}.$$

The integral relative to u must be taken from $u = 0$ to $u = 1$; but, at the maximum of the differential function $yy'\,dx\,dx'$, we have

$$x = a \qquad \text{and} \qquad x' = a'$$

and, consequently,

$$u = \frac{a'}{a}.$$

The value of u, corresponding to this maximum, exceeds therefore very slightly unity; thus we must, in this case, make use of formula (*c*) of No. VI. Let

$$du' = \frac{du}{\sqrt{-\frac{d^2\mathrm{Z}}{\mathrm{Z}dx^2}}},$$

and we name Z′ that which Z becomes at the point where we have

$$0 = \frac{\partial \mathrm{Z}}{\partial u'};$$

we name next S that which Z becomes when we make $u = 1$; the formula cited will give, quite nearly,

$$\int \mathrm{Z}\,du' = \frac{\mathrm{Z}'\int dt\,e^{-t^2}}{\sqrt{-\frac{1}{2}\,\frac{\partial^2\mathrm{Z}'}{\mathrm{Z}'\partial u'^2}}},$$

the integral relative to t being taken from $t = \mathrm{T}$ to $t = \infty$, T being given by the equation

$$\mathrm{T}^2 = \log \mathrm{Z}' - \log \mathrm{S}.$$

The equation $0 = \frac{\partial \mathrm{Z}}{\partial u'}$ can be put under this form

$$0 = \frac{\partial \mathrm{Z}}{\partial u} \frac{\partial u}{\partial u'},$$

whence we deduce

$$0 = \frac{\partial \mathrm{Z}}{\partial u}$$

and

$$\frac{\partial^2 \mathrm{Z}'}{\partial u'^2} = \frac{\partial^2 \mathrm{Z}'}{\partial u^2} \frac{du^2}{du'^2} = -\frac{\frac{\partial^2 \mathrm{Z}'}{\partial u^2} \frac{\partial^2 \mathrm{Z}'}{\partial x^2}}{\mathrm{Z}'};$$

we will have therefore

$$\int \mathrm{Z}\, du' = \frac{\mathrm{Z}'}{\sqrt{\frac{1}{2} \frac{\partial^2 \mathrm{Z}'}{\mathrm{Z}' \partial u^2} \frac{\partial^2 \mathrm{Z}'}{\mathrm{Z}' \partial x^2}}} \int dt\, e^{-t^2}.$$

The numerator of the expression of P will be, consequently, very nearly equal to

$$\frac{2\sqrt{\pi}\,\mathrm{Z}'}{\sqrt{\frac{\partial^2 \mathrm{Z}'}{\mathrm{Z}' \partial x^2} \frac{\partial^2 \mathrm{Z}'}{\mathrm{Z}' \partial u^2}}} \int dt\, e^{-t^2};$$

we will now determine its denominator.

y being null at the two limits $x' = 0$ and $x' = 1$, it is clear that z is null at the two limits $u = 0$ and $u = \frac{1}{x}$; it is similarly null at the two limits $x = 0$ and $x = 1$. By naming therefore U that which z becomes when we substitute for u and for x their values given by the equations

$$0 = \frac{\partial z}{\partial u} \qquad \text{and} \qquad 0 = \frac{\partial z}{\partial x},$$

we will have, by No. VII,

$$\int\!\!\int z\,du\,dx = \frac{2\pi\,\mathrm{U}}{\sqrt{\frac{\partial^2\mathrm{U}}{\mathrm{U}\partial u^2}\;\frac{\partial^2\mathrm{U}}{\mathrm{U}\partial x^2}}};$$

this is the value quite approximate of the denominator of P. It is easy to see that $\mathrm{Z}' = \mathrm{U}$, because the one and the other of these quantities is that which z becomes when we substitute for u and x their values deduced from the equations

$$0 = \frac{\partial z}{\partial u}, \qquad 0 = \frac{\partial z}{\partial x};$$

the value of P will be, consequently, given by this very simple formula

$$\mathrm{P} = \frac{\int dt\,e^{-t^2}}{\sqrt{\pi}}.$$

The two limits between which the integral relative to t must extend are $t = \mathrm{T}$ and $t = \infty$, T^2 being equal to $\log \mathrm{Z}' - \log \mathrm{S}$. The maximum of z or of xyy' is Z; the maximum of y corresponds to $x = a$; that of xy corresponds to a value of x which differs from it only by a quantity of order α, and as, at the point of the maximum, the magnitudes vary only in an insensible manner, we can suppose $x = a$ at the maximum of xy. Let Y be that which y becomes in this case, the maximum of xy will be aY. The maximum of y' corresponds to $x' = a'$; let Y' be that which y' becomes, we will have therefore $\mathrm{Z}' = a\mathrm{YY}'$. S is the maximum of xyy' when $u = 1$, or, that which returns to the same, when we make $x' = x$ in y'; let a'' be the value of x which in this case renders yy' a maximum, and we name Y'' this maximum, we will have $\mathrm{S} = a''\mathrm{Y}''$: hence

$$\mathrm{T}^2 = \log \mathrm{Y} + \log \mathrm{Y}' - \log \mathrm{Y}'' + \log \frac{a}{a''}.$$

The value of a'' is mean between a and a', and since these last two quantities are supposed to differ very little between themselves, we will have very nearly $\frac{a}{a''} = 1$, and consequently we can neglect the term $\log \frac{a}{a''}$.

If T^2 is a slightly large number, such as 11 or 12, P will be a very small fraction less than $\frac{1}{500000}$; it will be therefore hardly probable that the possibility of the simple event is greater in the first place than in the second, or, that which returns to the same, it will be very probable that, in the second place where a' surpasses a, the possibility of the simple events is greater than in the first. The observations will indicate then, with much likelihood, that there exists in the second place a cause of more than in the first, which facilitates the production of the simple event. The following analysis will give the law according to which this probability increases with the expansion of the simple events.

For this, we will observe that, a'' being very slightly different from a and from a', we will have quite nearly

$$\log \mathrm{Y}'' = \log \mathrm{Y} + \log \mathrm{Y}' + \frac{1}{2}(a''-a)^2 \frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2} + \frac{1}{2}(a''-a')^2 \frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2},$$

that which gives

$$\mathrm{T}^2 = \,-\,\frac{1}{2}(a''-a)^2 \frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2} - \frac{1}{2}(a''-a')^2 \frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2};$$

but a'' is given by the equation

$$0 = \frac{dy}{y\,dx} + \frac{dy'}{y'dx'},$$

x' needing to be changed into x in $\frac{dy'}{y'dx'}$. If we suppose next $x = a'' = a + (a''-a),$ we have

$$\frac{dy}{y\,dx} = \frac{d\mathrm{Y}}{\mathrm{Y}\,dx} + (a''-a)\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2};$$

moreover we have $0 = \frac{d\mathrm{Y}}{dx}$. We will have therefore

$$\frac{dy}{y\,dx} = (a''-a)\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2};$$

we will find similarly

$$\frac{dy'}{y'\,dx'} = (a''-a')\frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2}.$$

We will have therefore

$$0 = (a''-a)\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2} + (a''-a')\frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2},$$

whence we deduce

$$a'' = \frac{a \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2} + a' \frac{d^2 \mathrm{Y}'}{\mathrm{Y}'\, dx'^2}}{\frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2} + \frac{d^2 \mathrm{Y}'}{\mathrm{Y}'\, dx'^2}};$$

we will have thus very nearly

$$\mathrm{T}^2 = \frac{\frac{1}{2}(a' - a) \frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2} \frac{d^2 \mathrm{Y}'}{\mathrm{Y}'\, dx'^2}}{\frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2} + \frac{d^2 \mathrm{Y}'}{\mathrm{Y}'\, dx'^2}}.$$

We can easily judge, by this value of T^2, of the probability with which the observations indicate a difference between the possibilities of the simple events; because, this probability being, by that which precedes, equal to $1 - \frac{\int dt\, e^{-t^2}}{\sqrt{\pi}}$, the integral being taken from $t = \mathrm{T}$ to $t = \infty$, a Table of values of this integral, from $t = \infty$ to $t = 0$, will give immediately the sought probability with sufficient precision.

The simple events, by being expanded, make the values of $\frac{d^2 \mathrm{Y}}{\mathrm{Y}\, dx^2}$ and of $\frac{d^2 \mathrm{Y}'}{\mathrm{Y}'\, dx'^2}$ increase, and consequently also that of T^2, that which indicates clearly the law which exists between their expansion and the probability of the results which they seem to indicate. The value of T^2 shows further the more the differences between a and a' are smaller, the more simple observed events are necessary to establish that these differences are not the effect of chance, that which moreover is evident *a priori*, and there results from it that, for a difference two times less, there are necessary around four times more observations.

XXXIX.

We apply the formulas of the preceding sections to the births; for this we suppose that, out of $p + q$ observed births, there have been p boys and q girls, p being greater than q, and we seek the probability that the possibility of the births of the boys not surpass any quantity θ. It is necessary, in this case, to make use of the formulas of No. XXXVI. If we designate by x the possibility of the births of the boys and if we name β the quantity $\frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}$, the probability that out of $p + q$ births there will be p boys and q girls will be $\beta x^p (1 - x)^q$: it is the quantity which we have named y in the section cited; the quantity which we have named v will become thus $\frac{x(1-x)}{(p+q)x - p}$, and the function

$$\mathrm{UJ}\left(1+\frac{d\mathrm{U}}{d\theta}+\cdots\right)$$

will become

$$\frac{\beta\theta^{p+1}(1-\theta)^{q+1}}{(p+q)\theta-p}\left\{1-\frac{(p+q)\theta^2+p(1-2\theta)}{[(p+q)\theta-p]^2}+\cdots\right\}.$$

Now, the quantity which we have named U in No. XXXVI is, by No. VI, equal to $\sqrt{-\frac{2\mathrm{Y}dx^2}{d^2\mathrm{Y}}}$, Y and d^2Y being that which y and d^2y become when $x=a$; moreover, a being the value of x which corresponds to the maximum of y, it is determined by the equation $0=\frac{dy}{y\,dx}$, whence we deduce $a=\frac{p}{p+q}$, and consequently

$$\mathrm{Y}=\frac{\beta p^p q^q}{(p+q)^{p+q}},\qquad -\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2}=\frac{(p+q)^3}{pq}.$$

The function

$$\mathrm{Y}\sqrt{\pi}\left(U+\frac{1}{2}\,\frac{d^2U^3}{1.2\,dx^2}+\cdots\right)$$

will become therefore, by observing that it is reduced to very nearly its first term, when p and q are large numbers,

$$\frac{\beta p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}\sqrt{2\pi}}{(p+q)^{p+q+\frac{3}{2}}};$$

the formula (a') of the section cited will give thus for the probability that x not surpass θ

$$\frac{\theta^{p+1}(1-\theta)^{q+1}(p+q)^{p+q+\frac{3}{2}}}{\sqrt{2\pi}[p-(p+q)\theta]p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}}\left\{1-\frac{(p+q)\theta^2+p(1-2\theta)}{[p-(p+q)\theta]^2}+\cdots\right\}.$$

If we make $\theta=\frac{1}{2}$, we will have for the probability that x not surpass $\frac{1}{2}$ or, that which returns to the same, that the possibility of the births of the boys is less than that of the girls,

$$(e') \qquad \frac{(p+q)^{p+q+\frac{3}{2}}}{(p-q)2^{p+q+\frac{3}{2}}p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}\sqrt{\pi}}\left[1-\frac{p+q}{(p-q)^2}+\cdots\right];$$

by subtracting this formula from unity, we will have the probability with which the observed births indicate a greater possibility in the births of boys than in those of girls.

Among the births observed in Europe, we will consider those which have been at London, at Paris and in the realm of Naples.

In the space of the ninety-five years elapsed from the beginning of 1664 to the end of 1758, there is born at London 737629 boys and 698958 girls, that which gives nearly $\frac{19}{18}$ for the ratio of the births of boys to those of girls.

In the space of twenty-six years elapsed from the beginning of 1745 to the end of 1770, there is born at Paris 251527 boys and 241945 girls, that which gives $\frac{26}{25}$ nearly for the ratio of the births of boys to those of girls.

Finally, in the space of the nine years elapsed from the beginning of 1774 to the end of 1782, there is born in the realm of Naples, not containing Sicily, 782352 boys and 746821 girls, that which gives $\frac{22}{21}$ nearly for the ratio of the births of boys to those of girls.

The less considerable of these three numbers of births is that of the births observed at Paris; moreover it is in this city that the births of boys and of girls are removed the less from equality: for these two reasons, the probability that the possibility of the births of boys surpasses $\frac{1}{2}$ must be less than at London and in the realm of Naples. We will determine numerically this probability.

It is necessary for this to have to twelve decimals the tabulated logarithms of p, q, $p+q$ and 2, because these numbers are elevated in formula (e') to some great powers; now we have

$$\begin{aligned} \log p = &\quad \log 251527 = &&\log 5.4005\,8461\,0947,\\ \log q = &\quad \log 241945 = &&\log 5.3837\,1665\,1469,\\ \log(p+q) = &\quad \log 493472 = &&\log 5.6932\,6251\,5480,\\ & \quad\log 2 = &&\log 0.3010\,2999\,5664, \end{aligned}$$

that which gives

$$\log \frac{(p+q)^{p+q+\frac{3}{2}}}{(p-q)p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}2^{p+q+\frac{3}{2}}\sqrt{\pi}} = -41.9384918.$$

By naming therefore μ the number to which this logarithm belongs, and which is excessively small, because it is equal to a fraction of which, the numerator being unity, the denominator is the number 8 followed by 41 ciphers, the formula (e') will become

$$\mu(1 - 0.0053747 + \cdots).$$

By subtracting it from unity, we will have the probability that at Paris the possibility of the births of the boys surpasses that of the girls, whence we see that this probability differs so little from unity, that we can regard as certain that the excess of the births of the boys over those of the girls, observed at Paris, is due to a greater possibility in the births of the boys.

If we apply similarly formula (e') to the births of boys observed in the principle cities of Europe, we will find that the superiority in the births of the boys, compared to those of the girls, observed everywhere, from Naples to Petersburg, indicates a greater possibility in the births of boys, with a probability very near to certitude. This result seems therefore to be a general law, at least in Europe, and if, in some small towns where we have observed only a less considerable number of births, nature seems to deviate from it, there is every place to believe that this deviation is only apparent and that in the long run the observed births in these towns would offer, by multiplying themselves, a result similar to the one of the great cities. Many philosophers, deceived by these apparent anomalies, have sought the causes of phenomena which are only the effect of chance; that which proves the necessity to encourage similar researches, for that of the probability with which the phenomenon of which we just determined the cause is indicated by the observations: the following example will confirm this remark.

Out of 415 births observed during five years in the little town of Viteaux, in Bourgogne, there have been 203 boys and 212 girls, that which gives nearly $\frac{23}{22}$ for the ratio of the births of girls to those of boys. The natural order appears reversed here, because the births of the girls surpasses those of the boys; let us see with what probability these observations indicate a greater possibility in the births of girls.

p having been supposed greater than q, in the preceding formulas, it represents in this case the number of girls and q that of the boys; formula (e')

will give the probability that the births of the boys surpasses those of the girls; but, this formula being divergent, it is necessary to use formula (b') of No. XXXVI, and we will find, after all the reductions, that, if we make $y = \beta x^p(1-x)^q$ and $\theta = \frac{1}{2}$, it will become

$$\frac{\int dt\, e^{-t^2}}{\sqrt{\pi}} + \frac{(p-q)e^{-\mathrm{T}^2}}{3\sqrt{\frac{1}{2}\pi pq(p+q)}},$$

the integral being taken from $t = \mathrm{T}$ to $t = \infty$, T^2 being given by the equation

$$\mathrm{T}^2 = p\log p + q\log q - (p+q)\log\frac{p+q}{2},$$

in which the logarithms are hyperbolic. This formula is the expression of the probability that the possibility of the births of the boys carries it over that of the births of the girls; if we substitute, in the place of p and of q, their preceding values relative to the town of Viteaux, we will find 0.329802 for this probability; by subtracting it from unity, the difference 0.670198 will be the probability that at Viteaux the possibility of the births of the girls is superior to that of the births of the boys; this greater possibility is therefore indicated only with a probability of two against one, that which is much more feeble to counterbalance the analogy which leads us to think that at Viteaux, as in all the towns where we have observed a considerable number of births, the possibility of the births of boys is greater than that of the girls.

XL.

We have seen, in the preceding section, that the ratio of the births of boys to that of girls is around $\frac{19}{18}$ at London, while it is at Paris around $\frac{26}{25}$; this difference seems to indicate, in the first city, a possibility in the births of boys greater than in the second city. We determine with what likelihood the observations indicate this result.

This problem is a particular case of that which we have solved in No. XXXVIII; thus we make use of the formulas which we have given there; for this, it is necessary to know the quantities which we have named y and y'. Let p be the number of births of boys observed at Paris, q that of the births of girls, and x the possibility of the births of boys in that city; if we make

$$\beta = \frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q},$$

the probability of the result observed at Paris will be

$$\beta x^p (1-x)^q;$$

this is the quantity y.

If we name similarly p' the number of births of boys observed at London, q' that of the births of girls, and x' the possibility of the births of boys in that city; if we make next

$$\beta' = \frac{1.2.3\ldots(p'+q')}{1.2.3\ldots p'.1.2.3\ldots q'};$$

the probability of the result observed at London will be

$$\beta' x'^{p'} (1-x')^{q'};$$

this is the quantity y'.

By designating therefore by P the probability that at Paris the possibility of the births of boys is greater than at London, we will have, by No. XXXVIII,

$$\mathrm{P} = \frac{\int dt\, e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = \mathrm{T}$ to $t = \infty$. We see that which T becomes in the present case.

We have, by the section cited,

$$\mathrm{T}^2 = \log \mathrm{Y} + \log \mathrm{Y}' - \log \mathrm{Y}'' + \log \frac{a}{a''}.$$

Y is the maximum of y or of $\beta x^p (1-x)^q$; the value of x which corresponds to this maximum is $\frac{p}{p+q}$; this is the quantity which we have named a. We will have therefore

$$\mathrm{Y} = \frac{\beta p^p q^q}{(p+q)^{p+q}};$$

we will have in the same manner

$$Y' = \frac{\beta' p'^{p'} q'^{q'}}{(p'+q')^{p'+q'}}.$$

Y'' is the maximum of yy' when we make $x' = x$ in y', that which gives

$$yy' = \beta\beta' x^{p+p'}(1-x)^{q+q'};$$

the value of x corresponding to the maximum of this function is $\frac{p+p'}{p+p'+q+q'}$; this is the quantity which we have named a''. We will have thus

$$Y'' = \frac{\beta\beta'(p+p')^{p+p'}(q+q')^{q+q'}}{(p+p'+q+q')^{p+p'+q+q'}};$$

these values give

$$\begin{aligned} T^2 = {} & (p+p'+q+q'+1)\log(p+p'+q+q') \\ & - (p+p'+1)\log(p+p') - (q+q')\log(q+q') \\ & + (p+1)\log p + q\log q - (p+q+1)\log(p+q) \\ & + p'\log p' + q'\log q' - (p'+q')\log(p'+q'). \end{aligned}$$

Now we have, by the preceding section,

$$\begin{aligned} p &= 251527, & p' &= 737629, \\ q &= 241945, & q' &= 698958, \end{aligned}$$

whence we deduce, by tabulated logarithms,

$$\begin{aligned} \log p &= 5.4005\,8461\,0947, \\ \log q &= 5.3837\,1665\,1469, \\ \log(p+q) &= 5.6932\,6251\,5480, \\ \log p' &= 5.8678\,3798\,2735, \\ \log q' &= 5.8444\,5108\,0009, \\ \log(p'+q') &= 6.1573\,3193\,2083, \\ \log(p'+p) &= 5.9952\,6474\,1371, \\ \log(q+q') &= 5.9735\,4485\,3243, \\ \log(p+p'+q+q') &= 6.2855\,7058\,5161. \end{aligned}$$

By making use of these logarithms, we would have

$$\mathrm{T}^2 = 4.5357576;$$

but, these logarithms were tabulated, it is necessary, as we know, to multiply them by the number 2.3025851, in order to reduce them to hyperbolic logarithms; we will have therefore the true value of T^2 by multiplying the preceding by the same number, that which gives

$$\mathrm{T}^2 = 10.4439679.$$

This put, if we determine the integral $\int dt\, e^{-t^2}$ by formula (c') of No. XXXVI we will have

$$\mathrm{P} = 0.0000025422(1 - 0.047875 + 0.0068759 - \cdots).$$

The first three terms of this expression give

$$\mathrm{P} = 0.00000243797 = \frac{1}{410178}.$$

This value of P is a little too large; but, as, in taking one term more, we would have a value too small, without the impairment of $\frac{1}{250}$, we see that it is quite near, and that thus there are odds of more than 400000 against 1 that there exists at London a cause of more than at Paris, which facilitates the births of boys.

The numerical calculation of T^2 supposes that we have the tabulated logarithms of $p,\ q,\ p+q,\ p', q', \ldots$ to twelve decimals; the Tables of Gardiner, which are those of which we make the most use, contain the logarithms of the first 1161 numbers to twenty decimals, and we can conclude from it the logarithms of the superior numbers; but the calculation that this supposes is too long; we can supplement it quite simply by consideration of the expression of T^2, and to determine the value of this quantity without recourse to the logarithms of the numbers superior to 1161.

For this, we put it under this form

$$\begin{aligned}
\mathrm{T}^2 = (p+1)\log\frac{p}{p+q} &+ q\log\left(1-\frac{p}{p+q}\right)\\
&+ p'\log\frac{p'}{p'+q} + q'\log\left(1-\frac{p'}{p'+q}\right)\\
&- (p+p'+1)\log\frac{p+p'}{p+p'+q+q'}\\
&- (q+q')\log\left(1-\frac{p+p'}{p+p'+q+q'}\right).
\end{aligned}$$

If we make α vary by a very small quantity the ratio $\frac{p}{p+q}$ in the function

$$(p+1)\log\frac{p}{p+q} + q\log\left(1-\frac{p}{p+q}\right),$$

it will not change sensibly in value, because it becomes then

$$(p+1)\log\left(\frac{p}{p+q}+\alpha\right) + q\log\left(1-\frac{p}{p+q}-\alpha\right);$$

by reducing $\log\left(\frac{p}{p+q}+\alpha\right)$ and $\log\left(1-\frac{p}{p+q}-\alpha\right)$ into series ordered with respect to the powers of α, and by rejecting the quantities of order α which are not multiplied by the large numbers p and q, it is reduced to

$$(p+1)\log\left(\frac{p}{p+q}\right) + q\log\left(1-\frac{p}{p+q}\right).$$

This put, we will seek, by the method of continued fractions, the fraction which, having a denominator equal or less than 1161, most near to $\frac{p}{p+q}$; the difference of this fraction and of $\frac{p}{p+q}$ being only of order α, we can use this fraction in the place of $\frac{p}{p+q}$, and, as the Tables give with twenty decimals the logarithms of its numerator and of its denominator, so that the logarithms of the numerator and of the denominator of the new fraction which we have by subtracting the preceding from unity, we will have easily the tabulated value of

$$(p+1)\log\left(\frac{p}{p+q}\right) + q\log\left(1-\frac{p}{p+q}\right).$$

We will find in the same manner the tabulated values of the other parts of the expression of T^2; we will have thus the tabulated of T^2, and this expression, taking less of it, will be the tabulated logarithm of $e^{-\mathrm{T}^2}$; we will have next the true value of T^2 by multiplying the preceding by 2.3025851.

We can nearly always employ, without sensible error, the formula of No. XXXVIII

$$\mathrm{T}^2 = \frac{\frac{1}{2}(a' - a)\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2}\,\frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2}}{\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2} + \frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2}}.$$

and, as we have, in this case,

$$a = \frac{p}{p+q}, \quad a' = \frac{p'}{p'+q'},$$
$$-\frac{d^2\mathrm{Y}}{\mathrm{Y}\,dx^2} = \frac{(p+q)^2}{pq}, \quad -\frac{d^2\mathrm{Y}'}{\mathrm{Y}'\,dx'^2} = \frac{(p'+q')^2}{p'q'},$$

we will have

$$\mathrm{T}^2 = \frac{\left(\frac{p'}{p'+q'} - \frac{p}{p+q}\right)^2 (p+q)^3(p'+q')^3}{2p'q'(p+q)^3 + 2pq(p'+q')^3}.$$

If we apply this formula to the observed births at Paris and in the realm of Naples, it will be necessary to suppose

$$p = 251527, \quad q = 241945,$$
$$p' = 782352, \quad q' = 746821,$$

that which gives

$$\mathrm{T} = 2.7206;$$

we find then the probability P, that the possibility of the births of boys at Paris is greater than in the realm of Naples, equal to around $\frac{1}{100}$; it is therefore likely that there exists in this realm, as at London, a cause of more than at Paris, which facilitates the births of boys; but the probability with which it is indicated by the observations is considerably too small again in order to pronounce irrevocably on this object.

XLI.

We will consider now the probability of future events, taken from past events, and we suppose that, having observed a result composed of any number of simple events, we wish to determine the probability that a future result composed of the same events.

If we designate by x the possibility of the simple events, by y the probability corresponding to the observed result, and by z that of the future result, y and z being functions of x; if we name next P the probability of the future result, taken from the observed result, it is easy to conclude from No. XXXIV

$$\mathrm{P} = \frac{\int yz\,dx}{\int y\,dx},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

This formula contains the law according to which the past events influence on the probability of future events; we examine this influence in some particular cases. For this, we suppose that an urn contains an infinity of white and black balls, and that, after having drawn from it a white ball, we seek the probability of bringing forth a similar ball in the following drawing. If we name x the ratio of the white balls of the urn to the total number of balls, it is clear that x will be the probability, as much of the observed event as of the future event; we will have therefore

$$\mathrm{P} = \frac{\int x^2\,dx}{\int x\,dx} = \frac{2}{3},$$

that is that there are odds of two against one that we will bring forth in the second drawing a ball similar to that of the first drawing.

By supposing always that we have brought forth a white ball in the first drawing, if we seek the probability of bringing forth next n black balls, x will be the probability of the observed result, and $(1-x)^n$ that of the future result; we will have therefore then

$$\mathrm{P} = \frac{\int x(1-x)^n\,dx}{\int x\,dx} = \frac{2}{(n+1)(n+2)}.$$

If the white and black balls were equal in number in the urn, we would have $\mathrm{P} = \frac{1}{2^n}$; this value of P is less than the preceding when n is equal to or greater than 4; whence there results that, although the first drawing renders probable that the white balls are in greater number than the black, however the probability of bringing forth four black balls in the following four drawings is more considerable than if we would suppose the number of black balls equal to that of the white balls. This result, which seems paradoxical, leads to this that the probability of bringing forth n black balls is equal to the probability of bringing forth one of them, multiplied by the probability that having brought forth from it a first we will bring forth from it a second, multiplied further by the probability that having brought forth two from it we will bring forth from it a third, and thus in sequence; and it is clear that these partial probabilities always proceed by increasing and end by being reduced to unity when n is infinite.

XLII.

We suppose the observed result composed of a great number of simple events; let a be the value of x, which renders y a maximum; Y this maximum; a' the value of x which renders yz a maximum; Y' and Z' that which y and z becomes then; we will have very nearly, by No. VI,

$$\int y\,dx = \frac{\mathrm{Y}^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{d^2\mathrm{Y}}{dx^2}}},$$

$$\int yz\,dx = \frac{(\mathrm{Y}'\mathrm{Z}')^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{d^2(\mathrm{Y}'\mathrm{Z}')}{dx^2}}};$$

the expression of P of the preceding number becomes therefore

$$\mathrm{P} = \frac{(\mathrm{Y}'\mathrm{Z}')^{\frac{3}{2}}\sqrt{-\frac{d^2\mathrm{Y}}{dx^2}}}{\mathrm{Y}^{\frac{3}{2}}\sqrt{-\frac{d^2(\mathrm{Y}'\mathrm{Z}')}{dx^2}}}.$$

This expression will be very close if the observed result is quite composite.

If this result were composed of an infinity of simple events, the possibility of these events would be, by No. XXXVII, equal to that which renders the observed result most probable; we can therefore without sensible error calculate the probability of a future less composite result, by supposing the possibility of the simple events equal to that which renders the probability of a very composite event a maximum; but this supposition would cease to be exact if the future result were itself very composite. Let us see at what point we can make use of it.

The observed result being composed of a very great number of simple events, we suppose that the future result is much less composite; the equation which gives the value of a' corresponding to the maximum of yz is

$$0 = \frac{dy}{y\,dx} + \frac{dz}{z\,dx};$$

$\frac{dy}{y\,dx}$ is a very great quantity of order $\frac{1}{\alpha}$, and, since the future result is very little composite with respect to the observed result, $\frac{dz}{z\,dx}$ will be of a lesser order which we will suppose equal to $\frac{1}{\alpha^{1-\lambda}}$; thus, a being the value of x which satisfies the equation $0 = \frac{dy}{y\,dx}$, the difference between a and a' will be of order a^{λ}, and we can suppose

$$a' = a + \alpha^{\lambda}\mu.$$

This supposition gives

$$\mathrm{Y}' = \mathrm{Y} + \alpha^{\lambda}\mu\frac{d\mathrm{Y}}{dx} + \frac{\alpha^{2\lambda}\mu^2}{1.2}\frac{d^2\mathrm{Y}}{dx^2} + \cdots;$$

but we have $\frac{d\mathrm{Y}}{dx} = 0$, whence it is easy to conclude that $\frac{d^n\mathrm{Y}}{dx^n}$ is of order equal or less than $\frac{1}{\alpha^{\frac{n}{2}}}$; the term $\frac{\alpha^{n\lambda}\mu^n}{1.2.3\ldots n}\frac{d^n\mathrm{Y}}{dx^n}$ will be consequently of the order $\alpha^{n\left(\lambda-\frac{1}{2}\right)}$. Thus the convergence of the expression in series of Y' supposes $\lambda > \frac{1}{2}$, and in this case Y' is reduced to nearly Y.

If we name Z that which z becomes when we make $x = a$, we will be assured in the same manner that Z' is reduced to Z.

Finally we will prove, by a similar reasoning, that $\frac{d^2(\mathrm{Y}'\mathrm{Z}')}{dx^2}$ is reduced to very nearly $\mathrm{Z}\frac{d^2\mathrm{Y}}{dx^2}$; by substituting these values into the expression of P, we will have

$$\mathrm{P} = \mathrm{Z},$$

that is that we can in this case determine the probability of the future result, by supposing x equal to the value which renders the observed result most probable; but it is necessary for this that the future result be sufficiently little composite in order that the exponents of the factors of z are of an order less than the square root of the exponents of the factors of y; if this is not, it exposes the preceding supposition to some sensible errors.

If the future result is a function of the observed result, z will be a function of y, which we will represent by $\phi(y)$; the value of z which renders yz a maximum is in this case the same as that which corresponds the maximum of y; we will have thus $a' = a$, and, if we designate $\frac{d\phi(y)}{dy}$ by $\phi'(y)$, the expression of P will give, by observing that $\frac{d\mathrm{Y}}{dx} = 0$,

$$\mathrm{P} = \frac{\phi(\mathrm{Y})}{\sqrt{1 + \mathrm{Y}\frac{\phi'(\mathrm{Y})}{\phi(\mathrm{Y})}}}.$$

Let $\phi(y) = y^n$, so that the future event is n times the repetition of the observed event, we will have

$$\mathrm{P} = \frac{\mathrm{Y}^n}{\sqrt{n+1}}.$$

This probability, determined under the supposition that the possibility of the simple events is equal to that which renders the observed result the most probable, is equal to Y^n; we see thence that the small errors which result from this supposition are accumulated by reason of the simple events which enter into the future result and become very sensible when these events are in great number.

XLIII.

Since 1745, when we have begun to distinguish at Paris the births of boys from those of girls, we have constantly observed that the number of the first was superior to that of the second, that which can give place to research how much it is probable that this superiority will be maintained in the space of a century.

Let p be the observed number of births of boys at Paris; q that of the girls; $2n$ the annual number of births; x the possibility of the births of the boys. The binomial $(x + 1 - x)^{2n}$ gives by its expansion

$$x^{2n} + 2nx^{2n-1}(1-x) + \frac{2n(2n-1)}{1.2}x^{2n-2}(1-x)^2 + \cdots,$$

and the sum of the first n terms will be the probability that the number of boys will carry it away, each year, over that of the girls. We name z this sum; z^i will be the probability that this superiority will be maintained during the number i of consecutive years. Hence, if P designates the true probability that this will take place, we will have, by No. XLI,

$$\mathrm{P} = \frac{\int x^p dx\, z^i (1-x)^q}{\int x^p dx\, (1-x)^q},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

If we name a the value of x which corresponds to the maximum of $x^p z^i (1-x)^q$, and if we designate by Z, $\frac{d\mathrm{Z}}{dx}$, $\frac{d^2\mathrm{Z}}{dx^2}$ that which z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ become when we change x into a, we will have, by No. VI,

$$\int x^p z^i\, dx\, (1-x)^q = \frac{a^{p+1}(1-a)^{q+1}\mathrm{Z}^i\sqrt{2\pi}}{\sqrt{p(1-a)^2 + qa^2 + ia^2(1-a)^2\frac{d\mathrm{Z}^2 - \mathrm{Z}d^2\mathrm{Z}}{\mathrm{Z}^2 dx^2}}}.$$

z being the sum of the first n terms of the function

$$x^{2n}\left[1 + 2n\frac{1-x}{x} + \frac{2n(2n-1)}{1.2}\left(\frac{1-x}{x}\right)^2 + \cdots\right],$$

we have, by No. XXI,

$$z = \frac{\int \frac{u^{n-1}du}{(1+u)^{2n+1}}}{\int \frac{u^{n-1}du}{(1+u)^{2n+1}}},$$

the integral of the numerator being taken from $u = \frac{1-x}{x}$ to $u = \infty$, and that of the denominator being taken from $u = 0$ to $u = \infty$. Let $u = \frac{1-s}{s}$, this value of z will become

$$z = \frac{\int s^n ds (1-s)^{n-1}}{\int s^n ds (1-s)^{n-1}},$$

the integral of the numerator being taken from $s = 0$ to $s = x$, and that of the denominator being taken from $s = 0$ to $s = 1$; thence it is easy to conclude

$$\frac{dz}{z\,dx} = \frac{x^n(1-x)^{n-1}}{\int s^n dx (1-s)^{n-1}}, \qquad \frac{d^2z}{z\,dx^2} = \frac{dz}{z\,dx}\,\frac{n-(2n-1)x}{x(1-x)},$$

the integral being taken from $s = 0$ to $s = x$. By changing x into a, we will have the values of Z, $\frac{d\mathrm{Z}}{\mathrm{Z}dx}$, $\frac{d^2\mathrm{Z}}{\mathrm{Z}dx^2}$; all the difficulty is reduced therefore to determining a.

Its value is given by the equation

$$0 = \frac{p}{a} - \frac{q}{1-a} + i\frac{d\mathrm{Z}}{\mathrm{Z}\,dx},$$

whence we deduce, by substituting in the place of $\frac{d\mathrm{Z}}{\mathrm{Z}\,dx}$ its preceding value

$$a = \frac{p}{p+q} + \frac{ia^{n+1}(1-a)^n}{(p+q)\int s^n ds(1-s)^{n-1}},$$

the integral being taken from $s = 0$ to $s = a$; this is the equation after which it is necessary to determine a. For this, we will observe that, a being greater than $\frac{p}{p+q}$, it surpasses sensibly the value of s, which corresponds to the maximum of $s^n(1-s)^{n-1}$; thus, n being a great number, we can suppose, in the preceding equation, that the integral is taken from $s = 0$ to $s = 1$, that which gives, by No. VI,

$$\int s^n ds(1-s)^{n-1} = \frac{n^{n+\frac{1}{2}}(n-1)^{n-\frac{1}{2}}\sqrt{2\pi}}{(2n-1)^{2n+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^{2n}\sqrt{n}}.$$

The equation which determines a will become thus, very nearly,

$$a = \frac{p}{p+q} + \frac{ia^{n+1}(1-a)^n 2^{2n}\sqrt{n}}{(p+q)\sqrt{\pi}}.$$

In order to solve it, we will observe that a differs very little from $\frac{p}{p+q}$, so that, if we suppose $a = \frac{p}{p+q} + \mu$, μ being quite small, and we will have, in a quite close manner,

$$\mu = \frac{i\sqrt{n}2^{2n}\left(\frac{p}{p+q}\right)^{n+1}\left(\frac{q}{p+q}\right)^n}{(p+q)\sqrt{\pi}} e^{-n\mu\frac{(p+q)(p-q)}{pq}}.$$

Now, if we divide by 26 the sum of the births observed at Paris from 1745 to 1770, we will have, very nearly, 19000 for the annual number of births; we will suppose thus $n = 9500,\ i = 100;$ we have besides

$$p = 251527, \quad q = 241945.$$

The preceding equation will give therefore

$$\mu = 0.000157929\, e^{-738.144\, \mu},$$

whence we deduce

$$\mu = 0.00014222$$

and, consequently,

$$a = 0.5098509.$$

The radical

$$\sqrt{p(1-a)^2 + qa^2 + ia^2(1-a)^2 \frac{d\mathrm{Z}^2 - \mathrm{Z}d^2\mathrm{Z}}{\mathrm{Z}^2 dx^2}}$$

becomes, by substituting, in the place of $\frac{d^2\mathrm{Z}}{\mathrm{Z}\,dx}$, its value $\frac{d\mathrm{Z}}{\mathrm{Z}\,dx}\ \frac{n-(2n-1)a}{a(1-a)}$, and, in the place of $\frac{d\mathrm{Z}}{\mathrm{Z}\,dx}$, its value $\frac{(p+q)a-p}{ia(1-a)}$ or $\frac{(p+q)\mu}{ia(1-a)}$, given by the equation of the maximum,

$$\sqrt{p(1-a)^2 + qa^2 + \mu(p+q)\left[\frac{(p+q)\mu}{i} + (2n-1)a - n\right]} = 369.419;$$

moreover we have, very nearly,

$$a^p(1-a)^q = \left(\frac{p}{p+q}\right)^p \left(\frac{q}{p+q}\right)^q e^{-\frac{\mu^2(p+q)^2}{2pq}}$$

and

$$e^{-\frac{\mu^2(p+q)^2}{2pq}} = 0.980229;$$

we will have therefore

$$\int x^p z^{100} dx(1-x)^q = 0.000663199\sqrt{2\pi}\left(\frac{p}{p+q}\right)^p \left(\frac{q}{p+q}\right)^q \mathrm{Z}^{100}.$$

We have next by No. VI, by taking the integral from $x = 0$ to $x = 1$,

$$\int x^p dx(1-x)^q = \frac{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}} \sqrt{2\pi}}{(p+q)^{p+q+\frac{3}{2}}} = 0.000711634\sqrt{2\pi}\left(\frac{p}{p+q}\right)^p \left(\frac{q}{p+q}\right)^q,$$

whence we deduce

$$\mathrm{P} = 0.931938\mathrm{Z}^{100},$$

so that there is no more but the question of having Z.

We have

$$\mathrm{Z} = \frac{\int s^n dx(1-s)^{n-1}}{\int s^n dx(1-s)^{n-1}},$$

the integral of the numerator being taken from $s = 0$ to $s = a$, and that of the denominator being taken from $s = 0$ to $s = 1$; it is easy to conclude from it that, if we make $1 - s = s'$, we will have

$$\mathrm{Z} = 1 - \frac{\int s'^{n-1} ds'(1-s')^n}{\int s'^{n-1} ds(1-s')^n},$$

the integral of the numerator being taken from $s' = 0$ to $s' = 1 - a$, that of the denominator being taken from $s' = 0$ to $s' = 1$. We will have thus, quite nearly, by No. VI,

$$\mathrm{Z} = 1 - \frac{\int dt\, e^{-t^2}}{\sqrt{\pi}},$$

the integral relative to t being taken from $t = \mathrm{T}$ to $t = \infty$, T being given by the equation

$$\mathrm{T}^2 = (n-1)\log\frac{1}{2(1-a)} + n\log\frac{1}{2a},$$

these logarithms being hyperbolic. We can give to this expression of T^2 this quite close form

$$\mathrm{T}^2 = (n-1)\log\frac{p+q}{2q} + n\log\frac{p+q}{2p} + \frac{n\mu(p+q)(p-q)}{pq},$$

and we will deduce from it

$$\mathrm{T}^2 = 3.66793.$$

If we make use of formula (c') of No. XXXVII, we will have

$$\int dt\, e^{-t^2} = \frac{e^{-\mathrm{T}^2}}{2\mathrm{T}}(1 - 0.136317 + 0.055747 - 0.037996 + 0.036256 - \cdots).$$

This series is quite convergent, but it has the advantage of giving alternatively a sum greater and lesser than the truth, according as we are stopped at a number of terms even or odd; by adding therefore to the sum of the first four terms the half of the fifth, the error will be less than this half and, consequently, below $\frac{1}{50}$ of the entire sum; we will have thus

$$\int dt\, e^{-t^2} = \frac{e^{-\mathrm{T}^2}}{2\mathrm{T}} 0.899562,$$

that which gives

$$\mathrm{Z} = 0.9966174$$

and, consequently,

$$\mathrm{P} = 0.664;$$

there is therefore, very nearly, odds of two against one that, in the space of a century, the births of boys will carry away each year at Paris over those of girls.

XLIV.

The preceding researches suffice to show the advantages of the analysis exhibited at the beginning of this Memoir, in the part of the theory of chances, where the question is to carry up from observed events to their respective possibilities and to determine the probability of future events. This analysis is not less useful in the solution of the problems where we seek the probability of a result formed of a great number of simple events, of which the possibilities are known: in order to give an example, we will suppose that we propose to have the probability that all the tickets in a lottery composed of n tickets, and of which there is extracted from it one at each drawing, all will be extracted after the number i of drawings.

I have given, in Volume VI of the *Mémoires des Savants étranges*[1], the solution of this problem, whatever be the number of tickets which we bring forth at each drawing, and there results from this that, in the case where there exit at each drawing only a single ticket, if we name y_i the probability that all the tickets will be extracted after the number i of drawings, we will have

$$y_i = \frac{\triangle^n s^i}{n^i},$$

the characteristic $\triangle$ being that of the finite differences, and s being necessary to suppose null in the final result. This expression, quite simple in appearance, would lead to some impractical calculations if n and i were very large numbers; it would be much more difficult yet to conclude from it the number i, to which corresponds a given value of y_i; but we can easily determine this number by the formulas of No. XXVII.

Formula (μ') of this section gives, very nearly,

$$\triangle^n s^i = \frac{\left(\frac{i}{a}\right)^{i+1} e^{sa-1}(e^a-1)^n}{\sqrt{\frac{i(i+1)}{a^2} - \frac{nie^a}{(e^a-1)^2}}}\left(1 + \frac{15l'^2 - 12ll''}{16l^3} + \frac{1}{12i}\right),$$

a, l, l', l'' being given by the following equations

$$\begin{aligned}
0 &= \frac{i+1}{a} - s - \frac{ne^a}{e^a-1},\\
l &= -\frac{i+1}{2a^2} - \frac{n}{2}\frac{e^a}{e^a-1} + \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^2,\\
l' &= -\frac{i+1}{3a^2} + \frac{n}{6}\frac{e^a}{e^a-1} - \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^2 + \frac{n}{3}\left(\frac{e^a}{e^a-1}\right)^3,\\
l'' &= -\frac{i+1}{4a^2} - \frac{n}{24}\frac{e^a}{e^a-1} + \frac{7n}{24}\left(\frac{e^a}{e^a-1}\right)^2 - \frac{n}{2}\left(\frac{e^a}{e^a-1}\right)^3 + \frac{n}{4}\left(\frac{e^a}{e^a-1}\right)^4.
\end{aligned}$$

If we suppose $s = 0$ and e^a of order n or i, these equations will become

[1] Oeuvres de Laplace, t. VIII, p.17. Mémoire sur les suites récurro-récurrentes et sur leurs usages dans la théorie des hasards.

$$a = \frac{i+1}{n}(1 - e^{-a}),$$
$$l = -\frac{i+1}{2a^2}, \quad l' = -\frac{i+1}{3a^3}, \quad l'' = -\frac{i+1}{4a^4};$$

the preceding formula will give therefore, in this case,

$$\frac{\triangle^n s^i}{n^i} = \frac{\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}} e^{na-i}(1 - e^{-a})^{n-i}}{\sqrt{1 - \frac{i+1-na}{n}}}.$$

Now we have

$$\left(\frac{i}{i+1}\right)^{i+\frac{1}{2}} = e^{-1},$$

and if we make $e^{-a} = z$, z being supposed a very small fraction of order $\frac{1}{i}$, we will have

$$(1 - e^{-a})^{n-i} = e^{(i-n)z}\left(1 + \frac{i-n}{2}z^2\right);$$

we have next

$$i + 1 - na = (i+1)z.$$

We will have therefore, very nearly,

$$\frac{\triangle^n s^i}{n^i} = e^{-nz}\left(1 + \frac{i - 2n + 1}{2n}z + \frac{i-n}{2}z^2\right) = y_i.$$

In order to determine z, we will observe that the equation $a = \frac{i+1}{n}(1 - z)$ gives, for the first value of a,

$$a = \frac{i}{n};$$

by designating therefore $e^{\frac{-i}{n}}$ by q, we will have, for a first value of z,

$$z = q;$$

this value substituted into the expression of a gives, for the second value of this quantity,

$$a = \frac{i+1}{n} - \frac{i}{n}q;$$

by substituting it into the equation $z = e^{-a}$, we will have, for the second value of z,

$$z = q\left(1 - \frac{i}{n} + \frac{i}{n}q\right),$$

whence it is easy to conclude

$$y_i = e^{-nq}\left(1 + \frac{i}{2n}q - \frac{i+n}{2}q^2\right).$$

This value of y_i will be very close if, n and i being very large numbers, q is of order $\frac{1}{n}$; and this is that which will always take place when y_i will not be a very small fraction, because then e^{-nq} will not be a very small number, that which supposes q of order $\frac{1}{n}$.

Let $y_i = \frac{1}{2}$, and we seek the number i of drawings to which this probability corresponds. We will have, in order to determine it, the following two equations

$$q - \frac{i}{2n^2}q + \frac{i+n}{2n}q^2 = \frac{\log 2}{n},$$
$$i = -n \log q,$$

these logarithms being hyperbolic.

Let $n = 10000$, we have

$$\text{log hyperb. } 2 = 0.6931472;$$

the first of the two preceding equations gives, for the first value of q, by neglecting the terms $-\frac{i}{2n^2}q$ and $\frac{i+n}{2n}q^2$,

$$q = 0.00006931472.$$

This value being of order $\frac{1}{n}$, we see that this is here the case of using the preceding expression of y_i. The second equation gives

$$i = 95768.5.$$

This value can differ yet by some units from the truth; but, by correcting the value of q by its mean, we will have

$$q = 0.00006932250,$$

that which will give, for the second value of i,

$$i = 95767.41;$$

whence it follows that there are odds a little less than one against one that all the tickets will exit after 95767 drawings, and that there are odds a little more than one against one that they will exit after 95768 drawings.