

Frege's Logicism and the Significance of Interpretive Analysis

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1 Introduction

The history of Western philosophy, at least as reflected in university curricula in the English-speaking world, is standardly divided into three major periods: ancient philosophy, early modern philosophy, and modern philosophy. Ancient philosophy is seen as inaugurated by Socrates, and the subsequent writings of Plato and Aristotle undoubtedly set much of the agenda for the development of Western philosophy. There were thinkers before Socrates – the so-called Pre-Socratics; but as their name suggests, their significance lies in their being forerunners to Socrates. Early modern philosophy is typically taken to start with Descartes; certainly, it is with Descartes that most university courses on early modern philosophy – and indeed, many introductory courses in philosophy – begin. Early modern philosophy may or may not be divided into 'rationalism' and 'empiricism', with individual courses sometimes carved up accordingly; but the period is generally regarded as culminating in the critical philosophy of Kant, who at least saw himself as synthesizing rationalism and empiricism. The importance of Kant is reflected in the courses that focus entirely on his philosophy, and indeed, on just his first *Critique*, the *Critique of Pure Reason*.

There is less of a consensus as to when 'modern' philosophy begins. Thirty or forty years ago, 'modern' philosophy may have been seen as starting with Descartes, but 'early modern' is the term that is now preferred for the period just mentioned, leaving 'modern' to refer more specifically to philosophy *after* Kant – or in some cases, *from* Kant, Kant's work being taken to mark the beginning of modern philosophy proper. In other cases, however, modern philosophy is not really seen as emerging until the turn of the twentieth century, leaving the nineteenth century in a somewhat anomalous position – in either a separate category of its own or dissolved into a variety of other classifications.¹

¹ There is a *Cambridge History of Seventeenth-Century Philosophy* (1998), a *Cambridge History of Eighteenth-Century Philosophy* (2006), and a *Cambridge History of Nineteenth-Century Philosophy*

Following on from courses on early modern philosophy and Kant in a university syllabus, one often finds courses offered on 'post-Kantian philosophy', but this tends to be categorized as 'continental' philosophy, with figures such as Hegel, Schopenhauer, Kierkegaard and Nietzsche covered. If we restrict ourselves to the nineteenth century, there are far fewer courses on 'non-continental' philosophy, although Mill will be discussed in courses on ethics and James and Peirce in courses in the States on pragmatism. Bradley and other British idealists are rarely studied in an undergraduate (or taught postgraduate) degree, even as 'honorary' continental philosophers.

Once we come to the twentieth century, however, there is far more agreement that we are into 'modern' (or 'contemporary') philosophy. Here what is dominant, at least in the English-speaking world, but increasingly elsewhere, is 'analytic' philosophy. Courses in analytic philosophy – or philosophy of language, philosophical logic, philosophy of mathematics, 'analytic' metaphysics, contemporary epistemology or modern philosophy of mind, may well begin with Frege, Russell and/or Moore. While Kant and British idealism are often recognized as the target of early analytic philosophy, and Kant, in particular, is often introduced first, if only to provide some background, there is little real engagement with the work of the nineteenth-century philosophers that bridge the historical gap between Kant, on the one side, and Frege, Russell and Moore, on the other side.

Until recently, Frege, Russell and Moore – as well as later analytic philosophers from Wittgenstein and Carnap onwards – have been treated as contemporaries, concerned with the same questions that occupy analytic philosophers today, their ideas taken as lying at the foundations of current projects and debates. Over the last decade, however, the history of analytic philosophy has become conceptualized as a 'period' in the history of philosophy to rival interest in ancient philosophy and early modern philosophy. There has been particular focus on early analytic philosophy – roughly, from 1879, when Frege's *Begriffsschrift* was published, to 1939, when the Second World War intervened. One of the main motivations for this concern has been the increasing debate as to what exactly 'analytic philosophy' means and the growing recognition of the differences

(forthcoming), for example; but the next one in the series is the *Cambridge History of Philosophy 1870-1945* (2003).

between the assumptions and projects of the early analytic philosophers and those of later analytic philosophers. For a long time, analytic philosophy has also been opposed to 'continental' philosophy, or more specifically (and more accurately) to the individual traditions that make up what people have (misleadingly) called 'continental' philosophy, such as phenomenology and hermeneutics. In recent years, too, however, the common origins of analytic philosophy and phenomenology, in particular, have been stressed, and the desire to explore the relationships and foster dialogue between the various twentieth-century traditions has also been a factor in the burgeoning interest in the history of analytic philosophy.

Whether or not this division into ancient philosophy, early modern philosophy and modern philosophy, with analytic philosophy now seen as the dominant tradition, is justified either intellectually or pedagogically, it is undoubtedly the case that advances in mathematics played a major role in the initial developments in the three relevant periods. In the case of ancient philosophy, Greek geometry inspired both Plato, as the *Meno* and *Republic* show, in particular, and Aristotle, in his work on logic.² In the case of early modern philosophy, Descartes' achievements in analytic geometry provided the model for his thinking about methodology in philosophy, as Descartes himself attests.³ In the case of both early analytic philosophy, as represented in the work of Frege and Russell, and phenomenology, as represented in the work of Husserl, concern with the foundations of mathematics was crucial. It is this latter concern, and its role in the development of analytic philosophy, upon which I want to focus in the present paper. First, however, to provide some context, I shall make some brief general comments about the influence of mathematics on philosophy.

2 The influence of mathematics on philosophy

Mathematics has exercised a powerful hold on the philosophical imagination throughout the history of philosophy, and not just at key points in that history. In broadest terms, it has provided both models of methodology and test cases for philosophical positions, and

² For discussion of these influences, see e.g. Beaney 2009; Byrne 1997; Menn 2002.

³ See e.g. Descartes, *Rules for the Direction of the Mind*, in *PW*, I, 16-17; 'Second Set of Replies', in *PW*, II, 110-11.

has continually given rise to philosophical questions. Philosophers from Plato and Aristotle onwards have been inspired by Euclidean methodology, for example; metaphysical and epistemological theories have had to confront the issue of the nature of mathematics and our knowledge of it; and there has been a whole host of specific developments in mathematics that have prompted philosophical debate, such as the introduction of new kinds of numbers (negative, complex, transfinite, etc.), the increasing use of mathematics in scientific understanding of the world, and the emergence of mathematical logic and set theory.

Right from the beginning, mathematics has been seen as making steady, genuine progress, each success building on the achievements of the past. Its methods have seemed reliable and its results, once demonstrated, secure. There have been periods in which one might speak of revolutionary developments, but nothing approaching the 'paradigm shifts' that have arguably occurred in the natural sciences. The rationalist tradition in philosophy, which can be traced from Plato through Descartes and Leibniz to Frege and Gödel, has drawn its inspiration from mathematics; and even philosophers outside the tradition have found in mathematics models for comparison and have appreciated the need to give an account of the nature of mathematics and the status of mathematical propositions.

The certainty with which it seems we can know things in mathematics has been a key attraction for philosophers. This was especially so in the early modern period, when mathematical propositions came to be regarded as both necessary and knowable a priori. Even when mathematicians themselves began to worry about the foundations of their discipline in the nineteenth century, certainty was still a goal that was assumed to be attainable. Concern with rigorization of the calculus and the attempt to ground it in number theory played a crucial role in the origins of analytic philosophy, and traditional metaphysical and epistemological questions were involved here. What *are* numbers – the natural numbers, the rational numbers, the real numbers, and so on – and how do we apprehend them? The logicist project that Frege and Russell pursued – in their different ways – was intended to answer these questions, and there is no doubt that the project has metaphysical and epistemological dimensions. But logicism also raised semantic issues, which were no less important in the development of analytic philosophy. In offering the

logicist analyses and definitions that Frege and Russell did, questions about their semantic interpretation inevitably came to the fore, and it is these questions that have given analytic philosophy its distinctive character and provided the driving force in its evolution. It was not just the problems of existence and knowledge that were important, in other words, but also the problems of analysis and interpretation.

3 Analysis in analytic philosophy

It might seem obvious, from its very name, that analytic philosophy has placed particular emphasis on analysis, and that this emphasis is therefore central to its character. But analysis has always been at the heart of philosophical method, so if we are going to say that something new arose in analytic philosophy, or that there is something distinctive about the analytic tradition as it developed in the twentieth century, then we need to specify exactly what forms of analysis were introduced and explain how they differed or elaborated on earlier forms of analysis. Answering these questions has been the main aim of a project on which I have been working over the last decade, and I shall outline here the framework I have proposed for exploring conceptions of analysis in the history of philosophy, and indicate what I have suggested was particularly new and fruitful in analytic philosophy.⁴ It is in this context that we can then appreciate the role that concern with the foundations of mathematics played in the development of analytic philosophy.

On my account, it is useful to distinguish three main *modes* of analysis, as I have called them – the *regressive*, the *decompositional*, and the *interpretive* (or *transformative*). Perhaps all three might be described as aspects of “a process of isolating or working back to what is more fundamental by means of which something, initially taken as given, can be explained or reconstructed”, which is how I have characterized analysis in its broadest sense.⁵ But while the regressive mode involves working back to the principles, premises, causes, etc., by means of which something can be derived or explained, the decompositional mode involves identifying the elements and structure of something, and the interpretive mode involves ‘translating’ something into a particular

⁴ For details, see especially Beaney 2007b, 2007c, 2009. I have drawn on §1 of Beaney 2007c, in particular, in the paragraphs that follow.

⁵ Cf. Beaney 2009, introductory paragraph.

framework. These modes may be realized and combined in a variety of ways, in constituting specific conceptions or practices of analysis. Where one mode is dominant in a given conception, we may talk, for example, of the decompositional conception; but it should be stressed that in actual practices of analysis, all three modes are typically combined.

The regressive mode occupied centre-stage in the ancient period, at least if Pappus' classic account of analysis in ancient Greek geometry is a guide. In his *Mathematical Collection*, composed around 300AD, Pappus wrote: "In analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order."⁶ The conception articulated here has remained a core conception of analysis ever since. Filtered through discussions of Aristotelian methodology during the Renaissance, it found expression in the Port-Royal *Logic* of the seventeenth century, for example, and can also be seen illustrated in a paper that Russell wrote in 1907 entitled 'The Regressive Method of Discovering the Premises of Mathematics'.⁷

Although the decompositional mode is also exhibited in ancient Greek geometrical analysis, it rose to prominence during the early modern period, inspired by Descartes' work in analytic geometry. It achieved its philosophically most significant form in the decompositional conception of conceptual analysis developed by Leibniz and Kant. Central to Leibniz's philosophy was what can be called his *containment principle*: "in every affirmative true proposition, necessary or contingent, universal or singular, the notion of the predicate is contained in some way in that of the subject, *praedicatum inest subjecto*" (1973, 62). Analysis was then seen as the process of decomposing the subject concept into its constituent concepts until the containment of the relevant predicate is explicit, thereby achieving a proof of the proposition. Although Kant came to reject the generality of Leibniz's view, he accepted that containment held the key to what he called

⁶ Tr. in Hintikka and Remes 1974, 8.

⁷ For more on the regressive conception, see Beaney 2002, §1.1; 2009, §2.

'analytic' truths. A true proposition of the form ' A is B ' is 'analytic', on Kant's account, if and only if the predicate B is contained in the subject A .⁸

The decompositional conception of analysis has dominated philosophy in the modern period, from Descartes onwards. Although Kantian (along with Hegelian) philosophy was rejected by Russell and Moore in their early work, they retained the underlying conception of analysis. Indeed, their rebellion against British idealism was grounded on their endorsement of decompositional analysis as the primary method of philosophy. This endorsement can be seen as one characteristic feature of the 'analytic' tradition that they helped found. But precisely because decompositional analysis was not itself new, this is hardly sufficient to explain what was ground-breaking about analytic philosophy. On my view, it is the role played, instead, by interpretive or transformative analysis that is particularly distinctive of analytic philosophy, or at least, of one central strand in analytic philosophy, and it is the interpretive mode of analysis that came of age in early twentieth-century philosophy, encouraged by concern with the foundations of mathematics.

4 The interpretive mode of analysis

Interpretive analysis was not new in analytic philosophy. Indeed, it is implicit in all forms of analysis. For in attempting to analyze anything, we need first to interpret it in some way – drawing a diagram, for example, or 'translating' an initial statement into the privileged language of logic, mathematics or science – in order for the resources of a relevant theory or conceptual framework to be brought to bear. In Euclidean geometry, for example, a diagram is typically provided in order to see exactly what is to be demonstrated and to provide the basis for adding the auxiliary lines that are needed in the required construction or proof. Proclus recognized the importance of what I am here calling interpretive analysis when he divided the process of establishing a Euclidean Proposition into six parts, which he describes as follows:

⁸ Again, for more on the decompositional conception of analysis, see Beaney 2002, §1.2; 2009, §4.

... the enunciation [*protasis*] states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition [*ekthesis*] takes separately what is given and prepares it in advance for use in the investigation. The specification [*diorismos*] takes separately the thing that is sought and makes clear precisely what it is. The construction [*kataskeuē*] adds what is lacking in the given for finding what is sought. The proof [*apodeixis*] draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion [*sumperasma*] reverts to the enunciation, confirming what has been proved. (CEE, 159/203)

There is a great deal to say about this division, not least with regard to what counts as ‘analysis’ and what counts as ‘synthesis’ here. But the important point for present purposes is the description of ‘enunciation’ and ‘exposition’, the latter of which I take to involve drawing the relevant diagram and labelling it appropriately. Such exposition may be uncontroversial, in the sense that nothing illegitimate or problematic takes place, but it does deserve recognition.

The importance of recognizing preliminary interpretive analysis comes out when we turn to the case of analytic geometry. For what happens here is that geometrical problems are first ‘translated’ into the language of arithmetic and algebra in order to solve them more easily. Indeed, in the sixteenth and seventeenth centuries, algebra was specifically called an ‘art of analysis’, and while this phrase was used in deliberate allusion to the supposed secret art of analysis of the ancients, it is also appropriate, as I see it, in light of the role played by interpretive analysis. (Central to analysis in ancient Greek geometry was the idea of taking something as a ‘given’ and working back from there – an idea that is reflected in algebra in representing the ‘unknown’ to be found by ‘ x ’.⁹) At least on occasions, Descartes himself may have conceived of analysis in decompositional terms, as illustrated, for example, in his stressing how complex problems should be broken down into simpler ones.¹⁰ But that does not mean that interpretive analysis played no role in his practice. Indeed, the superior power of analytic geometry as opposed to synthetic geometry – as Euclidean geometry then came to be called by contrast – lies precisely in the translation of geometrical problems into arithmetical and algebraic ones, allowing the richer resources of arithmetic and algebra to be utilized.

⁹ Cf. Beaney 2009, §4.

¹⁰ See e.g. Descartes, *Rules for the Direction of the Mind*, in *PW*, I, 51; *Discourse on Method*, in *PW*, I, 120.

When we come to analytic philosophy, the significance of interpretive analysis is revealed most clearly in logical formalization. Just as in analytic geometry the problems are translated into the language of arithmetic and algebra to facilitate their solution, so too in analytic philosophy – or at least, in that central strand originating in the work of Frege and Russell – the statements seen as philosophically problematic are translated into the language of logic to reveal their ‘real’ logical form. If this analogy is right, then analytic philosophy is ‘analytic’ much more in the sense that analytic geometry is analytic than in the crude decompositional sense that many have taken ‘analysis’ to have.

I will discuss Frege's and Russell's use of interpretive analysis in the sections that follow. But let me note in passing that the essential strategy here – translating or paraphrasing a problematic statement into one which makes clearer what is meant, or what ontological commitments are involved, was not original in the work of Frege and Russell. It can be found in medieval discussions of so-called *exponibilia*,¹¹ for example, and in Bentham's conception of *paraphrasis*, which he characterized as “that sort of exposition which may be afforded by transmuting into a proposition, having for its subject some real entity, a proposition which has not for its subject any other than a fictitious entity” (*EL*, 246) and illustrated in ‘analyzing away’ talk of ‘obligations’.¹² What was crucial in the development of analytic philosophy was the invention of quantificational theory, which provided a far more powerful interpretive system than anything that had hitherto been available. The divergence that was thereby opened up between grammatical and logical form meant that the interpretive process itself became an issue of philosophical concern, inducing greater self-consciousness about our use of language and its potential to mislead. This inevitably raised questions about the relationships between language, logic, thought and reality which have been at the core of analytic philosophy ever since.

¹¹ Cf. Beaney 2009, §3.1.

¹² Cf. Beaney 2007b, 3; 2009, §6.

5 Frege's and Russell's use of interpretive analysis¹³

Frege's significance in the development of analytic philosophy lies in his invention of quantificational logic and the use he made of it in his logicist project – in seeking to show that arithmetic can be reduced to logic. The invention of quantificational logic made possible the formalization of far more complex propositions than had previously been logically analyzable. In particular, Frege was able to provide a logical analysis of number statements. His central idea here was that a number statement contains an assertion about a concept. A proposition such as 'Jupiter has four moons' is to be understood not as predicating of Jupiter the property of having four moons, but as predicating of the concept *moon of Jupiter* the second-level property *has four instances*, which can be logically defined. The significance of this construal can be brought out by considering negative existential statements (which are equivalent to number statements involving the number 0). Take the following negative existential statement:

(0a) Unicorns do not exist.

If we attempt to analyze this *decompositionally*, on the Leibniz/Kant model, taking its grammatical form to mirror its logical form, then we find ourselves asking what these unicorns are that have the property of non-existence. We may then be forced to posit the *subsistence* – as opposed to *existence* – of unicorns, just as Meinong and the early Russell did, in order for there to be something that is the subject of our proposition. On the Fregean account, however, to deny that something exists is to say that the relevant *concept* has no instances: there is no need to posit any mysterious *object*. The Fregean analysis of (0a) consists in *rephrasing* it into (0b), which can then be readily formalized in the new logic as (0c):

(0b) The concept *unicorn* is not instantiated.

(0c) $\neg(\exists x) Fx$.

Similarly, to say that God exists is to say that the concept *God* is (uniquely) instantiated, i.e., to deny that the concept has 0 instances (or 2 or more instances). On this view,

¹³ In this and the following two sections, I have taken over paragraphs, in a slightly revised form, from §§ 1 and 2 of Beaney 2007c.

existence is no longer seen as a (first-level) predicate, but instead, existential statements are analyzed in terms of the (second-level) predicate *is instantiated*, represented by means of the existential quantifier. As Frege notes, this offers a neat diagnosis of what is wrong with the traditional ontological argument (cf. 1884, §53). All the problems that arise if we try to apply decompositional analysis (at least straight off) simply drop away, although an account is still needed, of course, of concepts and quantifiers.

The eliminativist strategy that this form of analysis opens up was developed most famously in Russell's theory of descriptions. Here (Ka) is rephrased as (Kb), which can also then be readily formalized in the new logic as (Kc):

(Ka) The present King of France is bald.

(Kb) There is one and only one King of France, and whatever is King of France is bald.

(Kc) $(\exists x) (Kx \ \& \ (\forall y) (Ky \rightarrow y = x) \ \& \ Bx)$.

In interpreting (Ka) as (Kb), the definite description 'the present King of France' is 'analyzed away', so that any worries that might arise as to what the phrase means when there is no King of France disappear.

The possibilities that this strategy of 'translating' into a logical language opens up are enormous: we are no longer forced to treat the surface grammatical form of a statement as a guide to its 'real' form, and are provided with a means of representing that form. We become able to 'analyze away' problematic linguistic expressions and 'eliminate' the need to suppose that there are entities to which these expressions refer. But what is striking about Frege's use of interpretive analysis, despite its role in his account of number and existential statements, is his failure to appreciate its eliminativist potential. The idea of 'analyzing away' was never explicitly recognized as a methodological option.

Consider, for example, Frege's notorious problems with the paradox of the concept *horse*. On any natural view, the following proposition seems to be obviously true:

(Ha) The concept *horse* is a concept.

Yet *analyzing (Ha) compositionally*, the logically significant parts, on Frege's view, are the proper name 'the concept *horse*' and the concept expression '() is a concept'. If the proposition as a whole has a *Bedeutung*, then each of these parts must also have a *Bedeutung*, according to Frege. Since proper names stand for objects and concept expressions stand for concepts, and there is an absolute distinction between (unsaturated) concepts and (saturated) objects, 'the concept *horse*' must stand for an object, so that (Ha), taken literally, is false, not true. Clearly, something has gone wrong, and Frege's only response, biting the bullet, is to admit that 'the concept *horse*' does indeed stand for an object, but one that goes proxy for the concept, a response that seems as ontologically inflationary and metaphysically mysterious as the views of Meinong and the early Russell.¹⁴

In light of the idea of interpretive analysis, however, there is clearly a better response available. *(Ha) needs to be analyzed not compositionally, but paraphrastically*. And this is indeed just the response that Dummett (1981, 216-17) later made on Frege's behalf. On the assumption that the concept *horse* is sharp (i.e., that it divides all objects into those that fall under it and those that do not), (Ha) is to be interpreted as (Hb), which, like (0b) above, can be given a straightforward formalization in the predicate calculus, as (Hc):

(Hb) Everything is either a horse or not a horse.

(Hc) $(\forall x)(Hx \vee \neg Hx)$.

Given that the general strategy of analyzing by paraphrasing and logical formalization had been just what Frege had introduced, it may seem surprising that he failed to pursue it here, especially since the paradox of the concept *horse* seems to cry out for such treatment. But as the history of Russell's development between *The Principles of Mathematics* and 'On Denoting' shows, the possibility of using interpretive analysis to resolve ontological problems was a hard-won insight, and Frege never appreciated its potential.

¹⁴ Cf. Frege 1892/1997, 184-5. In accord with the editorial policy I adopted in Frege 1997, I have left '*Bedeutung*' untranslated.

6 Frege's understanding of abstraction principles

Frege's failure to appreciate the eliminativist potential of interpretive analysis was also responsible for the tension in his thought concerning the status of such principles as the Cantor-Hume Principle, to which he appealed in his *Grundlagen*, and Axiom V of the *Grundgesetze*, a tension which has generated controversy in the interpretation of Frege and in the recent debate over attempts to revitalize Frege's logicism. Consider the former, asserting the equivalence between the following two propositions:

(Na) The concept F is equinumerous to the concept G (i.e., there are just as many F s as G s).

(Nb) The number of F s is equal to the number of G s.

In the *Grundlagen* Frege clearly regards (Na) and (Nb) as having the same 'content' ('*Inhalt*'), but in his later work he vacillates somewhat between saying that such principles embody sameness of *Bedeutung* alone and saying that they embody sameness of both *Bedeutung* and sense (*Sinn*).¹⁵ His underlying thinking, however, seems to have been the following. If (Na) is true (and again, the point here is that (Na) is logically definable), and (Na) and (Nb) are (logically) equivalent, then (Nb) is true, that is, has a *Bedeutung*, on Frege's view (since the *Bedeutung* of a proposition just is its truth-value). But if this is so, then, by the principle of compositionality alluded to above, that the *Bedeutung* of a whole is dependent on the *Bedeutung* of its parts, all the logically significant parts of (Nb) also have a *Bedeutung*. So the number terms, as proper names, stand for independent objects.

Frege's use of the Cantor-Hume Principle suggests a method of defining abstract objects such as numbers contextually. But Frege did not himself see this as a method of *abstraction* – as others have understood it – in the sense of moving *up* an ontological level, from more to less basic objects. (Na) and (Nb) are regarded as on the same ontological level, an assumption that was responsible for the contradiction in Frege's system that Russell discovered in 1902. In seeking to explain or derive (Nb) from (Na),

¹⁵ For detailed discussion and references, see Beaney 1996, §§ 5.3 – 5.5, 8.1.

through interpretive (paraphrastic) analysis, and at the same time understanding (Nb) compositionally, Frege is trying to both have his cake and eat it. Insofar as (Nb) is genuinely equivalent to (Na), (Nb) cannot involve any other ontological commitments than are already involved in (Na), so (Nb) cannot be regarded as making reference to numbers construed as 'independent' objects. Rabbits can only be pulled out of hats if they are already there. So if the account of (Nb) runs through (Na), it cannot also be analyzed – ontologically – compositionally.

Of course, interpretive and compositional analysis are not in themselves incompatible. Indeed, in reductive projects, interpretive analysis gives way to compositional analysis once the problematic proposition has been rephrased into its correct logical form, where what counts as its correct logical form is governed by the purposes of analysis. If the aim is just to remove some philosophical puzzle (e.g., concerning the reification of non-existent entities), then interpretive analysis may be enough. But this will be unsatisfying to those who want an account of just what metaphysical commitments a proposition has. This raises the question of whether there can ever be an 'ultimate' analysis, however, an issue on which Frege, for whom function-argument analysis was central, and Russell, for whom whole-part analysis was fundamental, differed.

7 The significance of function-argument analysis in Frege's philosophy

Frege's most fundamental innovation was the use of function-argument analysis in developing his logical notation, an innovation that motivated all of Frege's characteristic doctrines, such as his distinction between concept and object and his conception of truth-values as objects.¹⁶ For current purposes what is of crucial importance is the idea, central to function-argument analysis, that two different functions can yield the same value for appropriate arguments. Applied to the case of propositions, this suggests that one and the same proposition (or its 'content') can be analyzed in different ways.

¹⁶ I discuss the role that function-argument analysis played in Frege's philosophy, with particular reference to his conception of truth-values as objects, in Beaney 2007d. See also Beaney 2011.

To appreciate the philosophical significance of this, consider the example that Frege gives in the *Begriffsschrift* (§9):

(HLC) Hydrogen is lighter than carbon dioxide.

According to Frege, this can be analyzed in (at least) two ways, depending on whether we take hydrogen as the argument and *is lighter than carbon dioxide* as the function, or carbon dioxide as the argument and *is heavier than hydrogen* as the function. If we respected subject-predicate position, we might wish to express the latter thus:

(CHH) Carbon dioxide is heavier than hydrogen.

But on Frege's view, (HLC) and (CHH) have the same 'content' (*Inhalt*), each merely representing alternative ways of analyzing that content. In reply, it might be suggested that both these analyses presuppose a more ultimate one, which identifies *two arguments*, hydrogen and carbon dioxide, and a *relation* (a function with two arguments). But which relation do we choose, *is lighter than* or *is heavier than*? Clearly they are not the same, since one is the converse of the other. So if we accept that (HLC) and (CHH) have the same 'content' (and there is undoubtedly something that they have in common), then it seems that there can be alternative analyses even at the supposedly ultimate level.

Let us compare Frege's view of analysis here with Russell's, at least as we have it at the time of the *Principles*. Russell would have regarded (HLC) and (CHH) as representing different propositions (as having different 'contents', in Frege's terminology), precisely on the grounds that there are two different relations involved here: "if we are to hold that "*a* is greater than *b*" and "*b* is less than *a*" are the same proposition, we shall have to maintain that both *greater* and *less* enter into each of these propositions, which seems obviously false" (1903, 228). What is driving this is the idea that a proposition is literally composed of what analysis yields as its constituents, and there is no room, so to speak, for a relational proposition to contain both the relevant relation and its converse.

How are we to decide the issue between Frege and Russell? Clearly, analysis is not as metaphysically neutral as the naïve idea of decomposition might suggest; it is not

just a matter of uncovering all those constituents that are there already, waiting to be uncovered. There are constraints on the process – in Frege's case, our judgements about sameness of 'content', and in Russell's case, the assumption that any complex whole, such as a proposition, is literally composed of its constituents. Frege never gave up the idea that two sentences could represent the same 'content', or express the same 'sense' or 'thought' as he later put it, even if they had different forms – as illustrated in the case of the Cantor-Hume Principle. Function-argument analysis allows this, since two different functions, with different arguments, can nevertheless yield the same value. Russell, on the other hand, never gave up the idea that complexes are literally composed of their constituents, even when the pressures of maintaining this with regard to propositions eventually led to his rejecting the very existence of propositions.¹⁷ Whole-part analysis was thus more deeply rooted in Russell's philosophy than in Frege's, even though, as we have seen, Frege assumed a decompositional conception himself at certain points in his thinking.¹⁸

As Frege understood abstraction principles, they afford a means of recognizing or identifying objects taken to already exist. It is significant here that Frege never called them 'abstraction' principles himself, despite noting Russell's talk of 'definition by abstraction'.¹⁹ For this might suggest that the relevant objects are indeed 'abstracted' out from the corresponding equivalence relation without already being there in the original domain. In grasping the sense of (Na), for example, and accepting the Cantor-Hume Principle, we thereby, according to Frege, apprehend the relevant objects that are the numbers. These objects are already there to be identified, and abstraction principles were simply seen as offering a way of doing so.

In commenting on the significance of function-argument analysis in his first work, *Begriffsschrift*, Frege wrote that "It is easy to see how taking a content as a function of an

¹⁷ Cf. Hylton 1996.

¹⁸ For more on the differences between Frege and Russell here, see Beaney 2003b, §6 (on which I have drawn in this section); Levine 2002.

¹⁹ See Frege's letter to Russell of 28 July 1902: Frege 1980, 141. In fact, the term is misleading as far as Russell is concerned, too. The principle of abstraction, Russell wrote in 1914, should really be called "the principle which dispenses with abstraction", since it "clears away incredible accumulations of metaphysical lumber" (1914, 51). For further discussion of Frege's and Russell's understanding of abstraction principles, see Beaney 2007b; Levine 2007.

argument gives rise to concept formation” (1879, vii/1997, 51). We can see this illustrated in the example of (HLC) above, which explicitly refers to the concept *is lighter than carbon dioxide*, but which can be analyzed differently to yield the concept *is heavier than hydrogen*. But Frege's understanding of abstraction principles suggests that he extended the basic idea to what might be called *object formation*. Splitting up the content of a proposition in a different way can also yield objects not explicitly referred to in its initial linguistic expression. In fact, however, ‘formation’ is not really the right word to use here to capture Frege's attitude. It would be better to talk of concept and object *recognition*, precisely because the relevant concepts and objects are taken to already exist: they are not brought into existence by analysis or abstraction. Such a view is what function-argument analysis encourages: in representing something as the value of a function for a particular argument, the object that is the argument is assumed as ‘given’ and a function is specified that maps that object onto the value. Function-argument analysis presupposes the existence of the relevant functions and arguments. Frege's philosophical attitude was staunchly realist, but that was rooted in his application of mathematical thinking.

8 Russell's paradox

In June 1902 Russell wrote to Frege informing him of the contradiction he had discovered in Frege's system, arising from his use of Axiom V. Despite a hastily-written appendix to the second volume of the *Grundgesetze* attempting to respond to it, Frege realized its seriousness and abandoned his logicist project shortly afterwards. Even in the first volume of the *Grundgesetze*, Frege had expressed some unease about the status of Axiom V,²⁰ which asserts the equivalence between the following two propositions:

- (Va) The function F has the same value for each argument as the function G .
- (Vb) The value-range (*Wertverlauf*) of the function F is identical with the value-range of the function G . (Cf. 1893, §§ 3, 9/1997, 213-14.)

²⁰ Cf. 1893, vii/1997, 195.

Since concepts are functions of one argument whose values are truth-values, on Frege's later view,²¹ (Va) and (Vb) yield the following as a special case (extensions of concepts being a type of value-range):

- (Ca) The concept *F* applies to the same objects as the concept *G* (i.e., whatever falls under concept *F* falls under concept *G*, and vice versa).
- (Cb) The extension of the concept *F* is identical with the extension of the concept *G*.

What we have here are abstraction principles, with exactly the same form as the Cantor-Hume Principle. As Frege saw it, Axiom V ensures that every (legitimate) concept has an extension in just the same way as the Cantor-Hume Principle guarantees that number terms have a *Bedeutung*.

Russell's paradox, in Fregean terms, can be stated as follows. If every concept is defined for all objects, then every concept can be thought of as dividing all objects into those that do, and those that do not, fall under it. If extensions of concepts are objects (as Frege had assumed they were, just like numbers), then extensions themselves can be divided into those that fall under the concept whose extension they are (e.g., the extension of the concept *is an extension*) and those that do not (e.g., the extension of the concept *is a horse*). But now consider the concept *is the extension of a concept under which it does not fall*. Does the extension of *this* concept fall under the concept or not? If it does, then it does not, and if it does not, then it does. Consider now the case in which the concept *F* and the concept *G* are one and the same. Then they have the same extension, so that (Cb) is true. But if this concept is the concept *is the extension of a concept under which it does not fall*, then it is not the case that anything that falls under this concept (the concept *F*) falls under this concept (the concept *G*), as the counterexample of its own extension shows, so that (Ca) is false. Axiom V, which asserts the equivalence between (Ca) and (Cb), is therefore false.

In the light of the discussion above, we might diagnose Frege's error as a failure to recognize that Axiom V is indeed an *abstraction* principle, which involves 'abstracting out' or 'constructing' objects (value-ranges, extensions of concepts) not there in the

²¹ See especially Frege 1891. I discuss the motivation for this view in Beaney 2007d.

original domain over which the equivalence relation is specified. On this view, extensions of concepts (classes) should be seen as on a higher ontological level than the objects which fall under those concepts (the members of the classes). This was the underlying idea of Russell's theory of types, which lay at the heart of his own attempt to resolve the paradox and take up the logicist mantle where Frege had abandoned it.

Frege was therefore right to have felt unease at the status of Axiom V, an unease reflected in his oscillation on the question of whether the two sides of Axiom V – (Va) and (Vb) – have the same sense or not. The dilemma for Frege might be put as follows. Either (Va) and (Vb) have the same sense, or they do not. If they do, then (Vb) cannot involve any other ontological commitments than are already involved in (Va), so (Vb) cannot be regarded as making reference to value ranges (including extensions of concepts) construed as 'proper' objects. On the other hand, if they do not, then how can Axiom V be regarded as a logical truth? If it does indeed make reference to 'proper' objects (legitimate arguments for all first-level functions), then Russell's paradox shows that far from being a logical truth, it is not even true.²²

9 The significance of interpretive analysis

The story of Frege's logicism, its abandonment as a result of Russell's paradox, and Russell's own logicism, is well-known, although the differences between Frege's and Russell's projects, rooted in their different philosophical attitudes, are less well appreciated. Frege and Russell are also widely recognized as two of the main founders of the analytic tradition, a tradition that places emphasis on analysis, although there has been a great deal of misunderstanding about the nature of analysis, which has too often been seen in crude decompositional terms. What I have wanted to bring out here is the connection between Frege's and Russell's concern with the foundations of mathematics and the development of analytic philosophy, a connection that goes far deeper than many have realized, partly as a result of failing to appreciate the significance of interpretive analysis.

²² For further discussion of Frege's and Russell's logicist projects, and the problems revealed in Frege's philosophy by Russell's paradox, see Beaney 2003b, 2005a.

I have focused in the last three sections on the case of abstraction principles, which have received a great deal of attention in recent years. We could equally well have taken the case of the logicist definition of the numbers as extensions of concepts or classes. Consider, for example, Frege's definition of the number 0 in the *Grundlagen*:

(E0) The number 0 is the extension of the concept *equinumerous to the concept 'not identical with itself'*.

Here, too, we have problems of analysis and interpretation. To what extent can such a definition be regarded as offering an 'analysis' of number? If we ask the person in the street what they mean by 'the number 0', they are highly unlikely to come up with the answer 'the extension of the concept *equinumerous to the concept "not identical with itself"*' or anything remotely like it. Does this mean that the senses of the *definiens* and *definiendum* should be taken as different? But if so, then how can this be regarded as a correct definition? What we have here is an illustration of the paradox of analysis, and there is much to be said about this, not just in relation to Frege's philosophy but also as a paradox that has been thought to threaten analytic philosophy itself.²³

Other examples could be given. When Russell remarked that "The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil" (1919, 71), he had in mind Dedekind's 'postulation' of the irrational numbers as limits of a series of ratios. Russell saw himself, by contrast, as actually 'constructing' them by defining them as classes, and the method of logical construction was highly influential – though controversial – in the second phase of analytic philosophy.²⁴ Frege's and Russell's logicist definitions were also the model for Carnap's conception of explication, a conception that in turn influenced Quine.²⁵

Problems of analysis, interpretation, construction and explication arise in a particularly stark form in the case of work on the foundations of mathematics. Concern with the foundations of mathematics was an essential feature of Frege's and Russell's philosophy, and we misunderstand that philosophy if we do this no justice. But equally,

²³ I have discussed this in detail in Beaney 1996, chs. 5, 8; 2005b.

²⁴ For discussion, see e.g. Linsky 2007.

²⁵ On Carnap's conception, see Beaney 2004; and on Quine's conception, see Hylton 2007, ch. 9.

the fundamental role that Frege's and Russell's work played in the development of analytic philosophy means that we must not lose sight of this concern in understanding analytic philosophy. On the contrary, the problems of analysis, interpretation, construction and explication lie at the heart of analytic philosophy, and these can be – and have been – fruitfully explored by considering questions concerning the foundations of mathematics.

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