the other way, by putting together identicals, the result runs perpetually into one and we never reach a plurality.

If we use 1 to stand for each of the objects to be numbered, we make the mistake of assigning the same symbol to different things. But if we provide the 1 with differentiating strokes, it becomes unusable for arithmetic.

The word "unit" is admirably adapted to conceal this difficulty; and that is the real, though no doubt unconscious, reason why we prefer it to the words "object" and "thing". We start by calling the things to be numbered "units", without detracting from their diversity; then subsequently the concept of putting together (or collecting, or uniting, or annexing, or whatever we choose to call it) transforms itself into that of arithmetical addition, while the concept word "unit" changes unperceived into the proper name "one". And there we have our identity. ... The difficulty is so well hidden under the word "unit", that those who have any suspicion of its existence must surely be few at most. (*FA*, §39.)

What this suggests is that 'one' and 'unit' function in quite different ways. Whilst 'one' (i.e. '1') appears to be what Frege calls a 'proper name' ('Eigenname'), 'unit' is a concept word. I consider the issues that this raises, as well as Frege's resolution of the difficulties he has identified in the various views he has examined, in the first section of the next chapter.

4. The Logicist Project

Do not permit yourself to think you have known truth in philosophy, unless you can explain the leap in which we deduce that one, two, three, and four together make ten. (St Augustine, as quoted by Leibniz, *LS*, p. 37.)

In the last chapter I sought to clarify the background to Frege's logicist project, sketching some of the developments in the history of mathematics, revealing the motivations in his early work, and outlining his criticisms of traditional conceptions of arithmetic. In this chapter I focus on his positive programme, elucidating informally its main details. I start by explaining his construal of number statements as containing assertions about concepts, and his belief that numbers are objects (§4.1); and then provide an exposition of his argument in §§62-9 of the *Grundlagen*, which is not only fundamental to his logicism but also pivotal in the development of his philosophy, as we shall see in the next chapter. In §4.3 I clarify the moves in his logical reconstruction of the natural numbers; and in §4.4 I outline his account of other numbers, introducing his *magnum opus*, the *Grundgesetze der Arithmetik*, which presents formally what he had merely sketched informally in the *Grundlagen*. In the final section, in commenting on its plausibility, I compare Frege's own theory with the theories of Cantor and Dedekind, in particular. The developments in Frege's philosophy which the *Grundgesetze* incorporates and anticipates will be the subject of later chapters.

4.1 Numbers, Concepts and Objects

In the course of his critique of rival conceptions of arithmetic in the first half of the *Grundlagen* (§§5-44), Frege established a number of (mainly negative) preliminary points. From our exposition in the previous chapter, we can summarize these as follows:

(a) Certain traditional arguments for construing arithmetical propositions as *synthetic* truths (whether *a priori* or *a posteriori*) are flawed (§§5, 7-10, 12). The only other possible option, that they are *analytic a priori* truths, remains viable (§§3, 6, 11, 15).

(b) Numbers are not properties of external things, since ascriptions of number depend on the concepts under which the things are classified

(§§21-5). What is numerable is everything thinkable, not just the sensible or intuitable (§§14, 24, 40).

(c) Numbers are not subjective ideas; they are objective, though non-actual (§§26-7).

(d) Numbers cannot be construed either as sets of objects or as sets of 'units'. The problem as to whether to treat units as the same or different shows that 'unit' ('Einheit') and 'one' ('eins') must be distinguished. (§§28, 34-9.)

Frege offers his own review in §45, and notes that the following question has remained unanswered: in making a statement of number (*Zahlangabe*), of what are we asserting something? He immediately proceeds to suggest the answer:

> To throw light on the matter, it will help to consider number in the context of a judgement that brings out its ordinary use. If, in looking at the same external phenomenon, I can say with equal truth 'This is a copse' and 'These are five trees', or 'Here are four companies' and 'Here are 500 men', then what changes here is neither the individual nor the whole, the aggregate, but rather my terminology. But that is only a sign of the replacement of one concept by another. This suggests as the answer to the first question of the previous section that a statement of number contains an assertion about a concept. This is perhaps clearest in the case of the number 0. If I say 'Venus has 0 moons', then there is no moon or aggregate of moons to assert anything of at all; but instead it is the concept 'moon of Venus' to which a property is ascribed, namely, that of including nothing under it. If I say 'The King's carriage is drawn by four horses', then I am ascribing the number four to the concept 'horse that draws the King's carriage'. (§46.)

In the rest of Part III of the *Grundlagen* Frege shows how this answer, treating a statement of number as asserting something about a concept, resolves the various issues he has identified. Taking the points we have just summarized in order, the following clarifications can then be offered:

(a) It is more plausible that assertions about concepts should be analytic. Whilst 'All whales are mammals', for example, might at first sight appear to be about animals (and hence be merely synthetic), in fact, Frege suggests in §47, it involves the subordination of concepts rather than the subsumption of objects under a concept. It should be construed, in other words, as 'The concept *whale* is subordinate to the concept *mammal*'. If we understand these concepts at all, we can then suggest, we will immediately recognize the truth of the proposition, which is to say that it is analytic (it is a matter of definition that a whale is a mammal).[1]

(b) We can explain how it might be thought that numbers are properties of external things, our ideas of both numbers and properties being abstracted from things themselves, just because we do in fact abstract concepts this way (though this is not the only way of acquiring concepts), the concepts then being what number statements are assertions about. We can also make sense of the universal applicability of number, since its domain is as wide as that of conceptual thought. (§§48-9.)

(c) To make assertions about concepts is not to say something about one's subjective ideas, since concepts are objective. It is an objective fact, for example, that the concept *whale* is subordinate to the concept *mammal*. (Cf. §47.)

(d) Something can only be called a 'unit' relative to a concept that precisely delimits what falls under it – what is now termed a *sortal* concept.[2] The concept *red*, for example, is *not* such a concept, since what falls under it (e.g. a surface) can be arbitrarily divided into parts all of which themselves fall under the concept. No finite number could therefore attach to this concept. The concept *moon of Jupiter*, on the other hand, *is* a sortal concept, each of the four moons of Jupiter being a unit relative to this concept. 'Units' are thus the *same* in so far as it is the same *concept* that is being applied, and *different* in so far as the *objects* numbered (falling under that concept) are different. (§54.)

Frege also suggests how his insight into the nature of number statements throws light on certain traditional problems relating to assertions of existence. The third of Frege's three 'fundamental principles' mentioned in his 'Introduction' to the *Grundlagen* is the principle that 'the distinction between concept and object must be kept in mind' (p. X), and this was a principle, as we saw in §2.4, that Frege was led to formulate as a result of his replacement of conventional subject-predicate analysis with function-argument analysis. As Frege remarked in his critique of Boolean logic, 'we must distinguish between concept and thing, even when only one thing falls under a concept' (*BLC*, p. 18). The point is stressed again in §51 of the *Grundlagen*: 'a concept does not cease to be a concept merely because only one single thing falls under it, which thing, accordingly, is fully determined by it'. Nor does a concept cease to be a concept if *nothing* falls under it, as Frege's earlier example of the concept *moon of Venus* shows (§46). What this suggests, then, is that *denial of existence* involves the assertion that the number 0 is assigned to the relevant concept, and *assertion of existence* involves the denial that the number 0 is assigned to the relevant concept (§53).

If existential statements are regarded, like number statements, as involving assertions about concepts, then, as Frege remarks, the ontological argument for the existence of God breaks down (ibid.). For existence is not a property of whatever it is that is supposed to fall under the concept *God*, that is, the concept of existence is not one of the 'characteristics' or 'marks' ('Merkmale') of the concept of God, and hence existence cannot be simply deduced by analysing the concept of God. Rather, the property of being instantiated is ascribed – rightly or wrongly – to the concept of God, and the truth of this ascription is not something that can be determined by definition. To say that God is omnipotent, for example, is to say that the concept of omnipotence is part of the concept of God; but to say that (the one true) God is existent is to say that the concept *God* is (uniquely) instantiated (i.e. that the number one is assigned to it). However, as Frege goes on to note, 'it should not be concluded that a property of a concept can never be deduced from the concept, that is, from its marks' (§53). For example, if we were to take seriously the stone paradox, that an omnipotent being cannot make a stone he cannot lift, then we might conclude that the concept of omnipotence is incoherent, and hence that the concept of God, which contains this concept, must be assigned the number 0.[3]

Denial of existence, then, need not be self-contradictory (as it would be in saying 'I do not exist'), and assertion of existence need not be uninformative (as it might be in saying 'These tame tigers exist'), and in general, existential statements need not be problematic, so long as what is involved are assertions about concepts – that certain concepts are or are not instantiated. But it remains the case that, for Frege, the use of *proper names* in logic *presupposes* the existence of objects. As we saw in §2.4, given his concept/object distinction and conception of logic, this was a natural position to adopt.[4] 'With a concept the question is always whether anything, and if so what, falls under it. With a proper name such questions make no sense.' (§51.) Whilst every sortal concept can be assigned a number from the whole range of natural numbers, in the case of proper names there is one and only one object involved.

By this point Frege has already indicated that he conceives number terms as proper names rather than concept words. In criticizing empiricism, he had rejected the view that number is a property of things, and in his discussion of 'unity' and 'one' he had added further arguments against so viewing the number one in particular (§§29-33). Firstly, since 'oneness' would presumably be a property possessed by everything, describing something as 'one' would say nothing at all. 'Only through the possibility of something not being wise does the assertion that Solon is wise gain a sense. The content of a concept diminishes as its extension grows; if the latter becomes all-embracing, then the content must be lost entirely.' (§29.) Secondly, if 'one' were a predicate, then 'Solon was one' would be just as legitimate as 'Solon was wise'. But 'Solon was one' is unintelligible on its own – without, say, 'wise man' being understood from the context. The point is even clearer in the plural case: 'Whilst we can combine "Solon was wise" and "Thales was wise" into "Solon and Thales were wise", we cannot say "Solon and Thales were one". The impossibility of this would not be perceived if "one" as well as "wise" were a property both of Solon and of Thales.' (Ibid.)

In §38 Frege makes explicit his two reasons, based on linguistic considerations, for believing that numbers are objects: firstly, that the definite article is used in constructions of the form 'the number one'; and secondly, that such number terms do not admit of plurals – we do not talk of several

number ones, for example (a point, as we saw in §3.5, that played a key role in his argument against construing numbers as sets of things). 'We say "the number one" and indicate by the definite article a definite and unique object of scientific study. There are not different numbers one, but only one. We have in 1 a proper name, which as such does not admit of a plural any more than "Frederick the Great" or "the chemical element gold".' (§38; cf. §51; §68, n.)

Of course, we are entitled to ask what significance these linguistic considerations have on questions of ontology. 'Frederick the Great', after all, no longer refers to an existent object; and fictional names too show that there is no necessary connection between the use of proper names and the existence of objects. The issues involved here are controversial, not least as a matter of Fregean exegesis, and I return to them later.[5] But the explanation (though not the justification) of Frege's belief should already be clear. Since logic, according to Frege, is a system of *truths*, proper names in logic *presuppose* the existence of objects. Since logic provides the framework for conceptual thought, and the realm of the conceptual is the realm of the enumerable, numbers have a central place in logic. If number terms are proper names rather than concept words, then they too must refer to objects. Given his assumptions, it is not surprising that Frege felt that all he had to establish was that number terms *are* proper names, which does seem to be merely a linguistic matter.[6]

However, Frege was aware that questions remained; and in the first subdivision of Part IV (§§55-61), he adduces two further linguistic considerations in support of his belief that numbers are objects. The first addresses the question that immediately arises in relation to his view that statements of number contain assertions about concepts. As we have seen, existential statements can be interpreted as statements involving the number 0, and Frege himself speaks loosely of existence as a *property* of concepts (§53), which suggests that number too is a property of concepts. But surely this is incompatible with the belief that numbers are *objects*? Frege's answer is provided in §57:

> In the proposition 'The number 0 belongs to the concept *F*', 0 is only a part of the predicate, if the concept *F* is taken as the real subject. I have therefore avoided calling a number such as 0, 1 or 2 a *property* of a concept. The individual number, by forming only a part of the predicate, appears precisely as an independent [*selbständiger*] object. (§57.)

Rephrasing Frege's example as 'The concept *F* is ascribed the number 0', making 'The concept *F*' the subject term, the predicate expression is then '( ) is ascribed the number 0', so that the property that is attributed to the concept *F* is not the number 0 itself but the property of *being ascribed the number 0*.[7] Compare this with 'Alfred is the adopted son of Gottlob', where Alfred is attributed the property of being the adopted son of Gottlob, which in no way prevents 'Gottlob' from representing an independent object. A proper name can be part of a predicate expression, in other words, without impugning its own semantic role.

Frege's second, though related, linguistic consideration concerns the *attributive* use of number terms. In 'Jupiter has four moons', 'four' is being used attributively, requiring qualification by the noun that follows.[8] But this type of construction too, Frege suggests, fails to undermine the belief that numbers are objects, since it can be rephrased to exhibit the number term as a proper name.

> For example, the proposition 'Jupiter has four moons' can be converted into 'The number of Jupiter's moons is four'. Here the 'is' should not be taken as a mere copula, as in the proposition 'The sky is blue'. This is shown by the fact that one can say: 'The number of Jupiter's moons is the number 4'. Here 'is' has the sense of 'is equal to' or 'is the same as'. We thus have an equation that asserts that the expression 'the number of Jupiter's moons' designates the same object as the word 'four'. And equations are the prevalent form of proposition in arithmetic. (Ibid.)

Both these considerations come together in the process of recasting statements of number into a form that makes explicit that number terms are proper names. For Frege's first example can be rephrased as 'The concept *F* has 0 instances', which we may call the *adjectival* construction, which in turn can be rephrased as 'The number of instances of the concept *F* is the number 0', the *substantival* construction, which Frege regards as the fundamental form.[9] The importance of the process of rephrasal in Frege's philosophy cannot be overemphasized. As we shall see, not only does it play a key role in his logicist project, but it also motivates his later reflections on meaning, for it is precisely the transitions involved here that require justification. All we need note at the moment, though, is that there seems to be, as yet, no reason to treat one type of construction as more primitive than the others, as the form to which the others are *reducible*. All that comes out of §57 is the idea that since equations (e.g. 4 = 2 + 2) are central in arithmetic, this is the fundamental form; but why should the reductive process not be reversed? Indeed, if statements of number are construed as assertions about concepts, surely it would be more natural to regard the adjectival construction as fundamental? Frege's argument does seem circular. Because numbers are independent objects, the substantival construction is fundamental; and because other constructions can be 'reduced' to this form, numbers are independent objects. I shall, however, return to these issues later.[10]

One final problem needs to be addressed here. For the obvious epistemological objection to the view that numbers are objects is that there seems to be no way of accounting for our apprehension of such objects. Frege rejects the supposition that we can have an *idea* of a number, since, as he has argued, 'it is neither something sensible nor a property of an external thing' (§58). And even if an idea (mental picture) is called up by hearing a word, 'the idea need not correspond to the content of the word; it may be quite different in different people' (§59). Furthermore, we can often make judgements about something, such as the Earth, or the distance of the Earth from the Sun, whilst having a very inadequate idea of it, or no idea at all (§§59-60). Frege goes on:

> That no idea can be formed of the content of a word is therefore no reason for denying it any meaning or for excluding it from use. The appearance to the contrary doubtless arises because we consider the words in isolation and in asking for their meaning, look only for an idea. A word for which we lack a corresponding mental picture thus appears to have no content. But one must always keep in mind a complete proposition. Only in a proposition do the words really have a meaning. The mental pictures that may pass before us need not correspond to the logical components of the judgement. It is enough if the proposition as a whole has a sense; its parts thereby also obtain their content. (§60.)

We have already discussed the first and third of the three 'fundamental principles' Frege singled out in his 'Introduction' to the *Grundlagen*. What we have here is the second, Frege's context principle: 'The meaning of a word must be asked for in the context of a proposition, not in isolation' (p. X). Its role is not just to provide support for the rejection of the supposition (orthodox in the Lockean tradition) that the meaning of a word is the idea it stands for, but also, more positively, to justify the assignment of meaning to number terms.

The context principle is one of the most influential yet controversial elements of Frege's philosophy. Despite the stress Frege places upon it in the 'Introduction' to the *Grundlagen*, it is not formulated again in his later work, unlike the other two principles; but nor is it explicitly repudiated. Part of the explanation of this is that Frege was still operating at the time of the *Grundlagen* with an undifferentiated notion of 'content', and this is later replaced by the dual notions of 'Sinn' and 'Bedeutung', which complicates any formulation of the principle; but it is also the case that Frege's realism increasingly gains control over his ideas, submerging his contextualism. I shall, however, postpone discussion of this until further elements of the picture are in place.[11] All we need note here is the use of the principle in removing one objection to Frege's construal of numbers as objects: we know what such objects are to the extent that we can use sentences in which number words occur. 'The independence that I am claiming for number is not to be taken to mean that a number word designates something when not in the context of a proposition, but I only intend by this to exclude the use of a number word as a predicate or attribute, which rather changes its meaning' (§60).

Of course, this still provides no reason for regarding substantival constructions as more fundamental than adjectival constructions; and even if we allow that there is some sense in which numbers can be treated as objects, this still leaves open the question as to exactly what kind of objects they are. For if we admit the context principle, why do we need to suppose that numbers are 'independent' ('*selbständige*') objects? Why can we not be content with viewing them as contextually defined abstract objects? Frege's objections to this suggestion emerge in the second subdivision of Part IV of the *Grundlagen* (§§62-9), an exposition of which is provided in the next section.

4.2 Frege's Central Argument

As we have now seen, the first three Parts of Frege's *Grundlagen* (§§5-54) consist in a critique of certain traditional views of arithmetic, and the difficulties in these views are resolved, according to Frege, by treating a statement of number as an assertion about a concept. It is this construal that suggests 'so naturally' the preliminary definitions that Frege offers in the very first section of Part IV (§55), the Part that contains the constructive argument of the *Grundlagen*.

(F_0) 'The number 0 belongs to a concept F' ('There are 0 F's') is defined as 'For all x, x is not F'.

(F_1) 'The number 1 belongs to a concept F' ('There is just 1 F') is defined as 'It is not true that, for all x, x is not F; and, for all x and y, if x is F and y is F, then $x = y$'.

(F_{n+1}) 'The number $n + 1$ belongs to a concept F' ('There are $n + 1$ F's') is defined as 'There is some x, such that x is F, and n is the number that belongs to the concept *falling under F, but not x*'.

Using the modern device of the *numerical quantifier*, '$\exists_n x$' being read as 'there are *n x's* such that', these can be formalized thus:

(F_0*) '$(\exists_0 x)\, Fx$' is defined as '$(\forall x)\, \neg Fx$'.
(F_1*) '$(\exists_1 x)\, Fx$' is defined as '$\neg(\forall x)\, \neg Fx\ \&\ (\forall x)\, (\forall y)\, (Fx\ \&\ Fy \rightarrow x = y)$'.
(F_{n+1}*) '$(\exists_{n+1} x)\, Fx$' is defined as '$(\exists x)\, [\, Fx\ \&\ (\exists_n y)\, (Fy\ \&\ x \neq y)\,]$'.

What we have here, it would seem, is a logical characterization of precisely the three number statements that enable the logicist project to get off the ground; and it comes as a surprise that Frege immediately rejects them as unsatisfactory. He does, however, rightly note that (F_{n+1}) is inadequate as it stands, 'for strictly speaking the sense of the expression "the number n belongs to the concept G" is just as unknown to us as that of the expression "the number $(n + 1)$ belongs to the concept F"' (§56); in other words, taking (F_{n+1}*), a definition of '$(\exists_{n+1} x)\, Fx$' is offered which makes use of '$\exists_n x$', which has not itself been defined. But rather than proceeding to supplement the account, he rephrases his objection to encompass what he feels is also wrong with the first two definitions,

namely, that 'we can never – to take an extreme example [*krasses Beispiel*] – decide by means of our definitions whether the number *Julius Caesar* belongs to a concept, or whether that well-known conqueror of Gaul is a number or not' (§56). The example does indeed strike most people as *krass*. But from what was said in the previous section, it should be clear what Frege feels is wrong with the definitions. We have been given no explicit definition of '*the* number which belongs to the concept F', leaving us with no way of determining whether the number a which belongs to the concept F and the number b which belongs to the concept G, say, are the same or not, i.e., for any a and b, whether $a = b$ or not.[12] Frege writes: 'It is only an illusion that we have defined 0 and 1; in truth we have only determined the sense of the phrases "the number 0 belongs to" [and] "the number 1 belongs to"; but this does not allow us to distinguish 0 and 1 here as independent, reidentifiable objects' (§56).

In the rest of the first subdivision of Part IV (§§57-61), as we saw in the last section, Frege proffers further support for his belief that numbers are independent objects. But as we commented, Frege's argument seems circular, and his linguistic considerations remain unconvincing. However, with the context principle formulated, Frege returns to the problem of providing adequate definitions in §62. Given that the context principle suggests a way of accounting for our apprehension of numbers without appealing to ideas or intuitions, 'It will therefore depend on defining the sense of a proposition in which a number word occurs'. He goes on:

> But we have already established that number words are to be understood as standing for independent objects. This gives us a class of propositions that must have a sense – propositions that express recognition [of a number as the same again]. If the symbol a is to designate an object for us, then we must have a criterion that decides in all cases whether b is the same as a, even if it is not always in our power to apply this criterion. In our case we must define the sense of the proposition
>
> > 'The number that belongs to the concept F is the same as the number that belongs to the concept G';
>
> that is, we must represent the content [*Inhalt*] of this proposition in another way, without using the expression
>
> > 'the number that belongs to the concept F'.
>
> In doing so, we shall be giving a general criterion for the identity [*Gleichheit*] of numbers. When we have thus acquired a means of grasping a definite number and recognizing it as the same again, we can give it a number word as its proper name. (§62.)

As §§55-6 had indicated, the task is to formulate definitions, without presupposing an understanding of the expression '*the* number that belongs to the concept F', which provide a means of determining, for any a and b, whether $a = b$ or not (equations being fundamental to arithmetic).

The strategy is to find a logically definable proposition, through which 'to form the content of a judgement that can be construed as an equation on each side of which is a number' (§63). The suggestion is to use the first of the following propositions to define the second:

(Na) The concept F is equinumerous to the concept G. (There are as many objects falling under concept F as under concept G, i.e. there are just as many F's as G's.)

(Nb) The number of F's is identical with the number of G's. (The number that belongs to the concept F is the same as the number that belongs to the concept G.)

The strategy is thus to define numerical identity (*Gleichheit der Zahlen*) in terms of one-one correlation (*beiderseits eindeutige Zuordnung*) or equinumerosity (*Gleichzahligkeit*).[13] Frege notes that this strategy 'seems recently to have gained widespread acceptance amongst mathematicians' (§63), but in the course of §§63-7 raises three objections to it, the first two of which, relating to the concept of identity, he answers, and the third of which he sustains.

The first objection concerns the specificity of the definition of numerical identity. 'The relationship of identity does not only hold amongst numbers. From this it seems to follow that it ought not to be defined specially for this case.' (Ibid.) However, he responds, his aim is not to define numerical identity in particular, but 'by means of the concept of identity, *taken as already known*, to obtain that which is to be regarded as being identical' (ibid.; my emphasis). Frege switches to a geometrical example to illustrate the method:

(Da) Line a is parallel to line b.

(Db) The direction of line a is identical with the direction of line b.

Symbolically, these may be represented thus:

(Da*) $a \,//\, b$.

(Db*) $\mathrm{Dir}(a) = \mathrm{Dir}(b)$.

By reconstruing (Da) with the help of the concept of identity, Frege argues, we can obtain the concept of direction: 'we replace the symbol // by the more general =, by distributing the particular content of the former to a and b. We split up the content in a different way from the original way and thereby acquire a new concept' (§64).

Frege's geometrical example raises problems of its own. As Frege recognizes, 'parallel lines are frequently defined as lines whose directions are identical', reversing what *he* sees as the relationship between (Da) and (Db). In response, Frege argues that since 'everything geometrical must surely originate in intuition', we can only start with an intuition of parallel lines, from which, 'through a mental act', we arrive at the concept of direction. (Ibid.) But this talk of intuition might itself be questioned, since

any dependence on a geometrical analogy may sully the purity of the logicist project.[14] However, it is wrong to read too much into the choice of example. Frege's concern is only to illustrate the *general form* of definition; and all we need to accept is the *possibility* of defining an abstract object such as a number or direction in terms of some equivalence relation defined over objects of some other kind, and the general concept of identity.

That it is legitimate to utilize the *general* concept of identity is reinforced by considering the second doubt that Frege raises, 'as to whether such a definition might not involve us in conflict with the well-known laws of identity' (§65). Frege takes as his definition of identity Leibniz's *salva veritate* principle:

(SV) *Eadem sunt, quorum unum potest substitui alteri salva veritate.* [Those things are the same of which one can be substituted for the other without loss of truth.]

What Frege understands by this is what is often called *Leibniz's Law* – interpreted as comprising both the Principle of the Indiscernibility of Identicals (reading the equivalence from left to right) and the Principle of the Identity of Indiscernibles (reading the equivalence from right to left):

(LL) $x = y \leftrightarrow (\forall F)\,(Fx \leftrightarrow Fy)$.[15]

What (LL) provides is a definition of identity in purely logical terms (within second order predicate logic, i.e. where quantification over properties is allowed); and it is this that justifies Frege in taking the concept of identity as already known. Frege goes on to remark that 'in universal substitutability all the laws of identity are contained' (§65), and this is correct, so long as the substitutability is restricted to extensional contexts, something that Frege was only clear about later.[16] But in this case, as he suggests (ibid.), every result of substituting 'the direction of b' for 'the direction of a' in an expression of a geometrical truth will itself be the expression of a geometrical truth; and this is to say that the direction of a is identical with the direction of b.

In §66, however, Frege raises his third objection, which he regards as unanswerable. According to Frege, as §56 had indicated, an adequate definition must enable us to *reidentify* the object introduced. But whilst the definition of (Db) by means of (Da) allows us to determine whether the direction of a given line is the same as that of any other given line, it does not tell us exactly *what* directions are, that is, what distinguishes them from any other objects, such as, to take Frege's second *krasses Beispiel*, England (§66). The general objection here, alluding to Frege's earlier example, has come to be known as 'the Caesar problem'.[17] The appropriate *criterion of identity* (as demanded in §62) has not been properly specified. An adequate definition must enable us to determine whether the following proposition is true *for any q*:

(Dq) The direction of line a is identical with q.

According to Frege, however, the suggested definition only enables us to do this 'if q is given in the form of "the direction of b" ' (§66). Any attempt to stipulate that (Dq) can only be true if q is a direction would presuppose the very concept of direction that we were trying to define; and any definition of the following form would be circular:

(q) q is a direction, if there is a line b whose direction is q. (Cf. §66.)

The only kind of non-circular definition that could be offered would be this:

(Dr) The direction of line a is whatever is in common to line a and any other line that is parallel to it.

Frege's point can then be put very simply: the phrase 'whatever is in common' is inadequate for singling out the kind of object that a direction is.

In §67, which is one of the important passages in the development of his views, Frege elaborates the problem further:

> If one were to say: q is a direction if it is introduced by means of the definition offered above [(q)], then the way in which the object q is introduced would be treated as a property of it, which it is not. The definition of an object asserts, as such, really nothing about it, but instead stipulates the meaning [*Bedeutung*] of a symbol. After this has been done, it transforms itself into a judgement which does deal with the object, but now it no longer introduces it but stands on the same level as other assertions about it. If this way out were chosen, it would presuppose that an object can only be given in one single way; for otherwise it would not follow, from the fact that q was not introduced by means of our definition, that it could not have been so introduced. All equations [identities] would then come down to this, that whatever is given to us in the same way is to be recognized as the same. But this is so self-evident and so unfruitful that it is not worth stating. Indeed, no conclusion could ever be drawn here that was different from any of the premisses. The multitude of meaningful [*bedeutsame*] uses of equations [identities] depends rather on the fact that something can be reidentified even though it is given in a different way.

In §24 of the *Begriffsschrift* Frege had talked of the *Doppelseitigkeit* of definitions, whereby an initial stipulation of the meaning of a symbol could be 'converted' into a corresponding judgement, and this has now merged with the distinction between introducing an object and asserting something *about* that object. If rephrased in terms of specifying a *Bedeutung* and expressing a *Sinn*, then the later distinction between *Sinn* and *Bedeutung* can be seen as emerging in the thinking out of the conception of the *Doppelseitigkeit* of definitions; and I shall return to this later (see especially §6.4). For the moment we need only note that what Frege realizes here is that in order for a definition to satisfy his fundamental

requirement, it is essential that the object introduced be recognizable in *other* ways than that involved in the initial definition. Only then can a definition transform itself into a judgement, and play a part in a scientific system (where *truths* are established). This leads Frege to the conclusion that the attempted definition of (Db) in terms of (Da) (and hence of any similar definition of number) is inadequate; and he goes on to consider a different way.

Frege's preferred way proceeds from the recognition that (Da) holds iff the following holds:

(Dd) The extension of the concept 'line parallel to line a' is identical with the extension of the concept 'line parallel to line b'.

He then suggests the following definition:

(De) The direction of line a is the extension of the concept 'parallel to line a'.

Having indicated the idea, he then drops the geometrical example, and provides a similar definition of number:

(Ne) The number that belongs to the concept F is the extension of the concept 'equinumerous to the concept F'. (Cf. §68.)

§68 of the *Grundlagen* contains one of Frege's most fundamental moves, crucial not only to his logicist project but also to the development of his philosophy; yet the argument is astonishingly compressed. Having rejected the contextual attempt to provide an adequate definition of *direction* by means of (Da) and (Db), Frege's alternative strategy is introduced without warning, and his proffered definition sprung on us with little explanation. All we are told is that (Da) and (Dd), and the corresponding two propositions, (Na) and (Nd), are logically equivalent:

(Na) The concept F is equinumerous to the concept G.
(Nd) The extension of the concept 'equinumerous to the concept F' is identical with the extension of the concept 'equinumerous to the concept G'.

(Nd) is true, Frege writes, iff (Na) is true (§69). But our remarks at the end of chapter 2 should make us cautious about placing any weight on considerations of logical equivalence at this stage of Frege's development; and I shall provide a more detailed analysis of the relationships involved here in §5.3.

But an obvious question arises straightaway. For how does Frege's alternative definition avoid the Caesar problem? Surely (Da) and (Db) are also logically equivalent, so how can (De) be an improvement on (Dr)? The answer, stated in the footnote to §68, but not justified anywhere in the *Grundlagen*, is that we are simply assumed to have knowledge of what the extension of a concept is.[18] Whilst (Dr) contains the vague phrase 'whatever is in common', Frege takes it that there is no such indeterminacy in (De). (De) is seen not only as stipulating a meaning for 'the direction of line *a*', but also as singling out the appropriate object in such a way that it *is* possible to reidentify it, just because we already know what the object is – the extension of the concept. Again, though, this is something to which we shall return in later chapters.

4.3 The Natural Numbers

Opening the third subdivision of Part IV of the *Grundlagen*, Frege writes: 'Definitions prove themselves by their fruitfulness' (§70). He has accepted that there may be doubts about his identification of numbers with extensions of concepts (cf. §69),[19] but so long as all 'the well-known properties of numbers' can be derived, he is satisfied that his definition can be justified (§70). As we saw in §3.3, what Frege regards as important for the status of a proposition is its 'most perfect method of proof' (*BS*, 'Preface'), and if a purely logical proof *can* be provided of every arithmetical proposition, then logicism is validated.

Let us first note that from Frege's explicit definition we can immediately derive the proposition that, according to Frege, had been inadequately defined contextually. For what we have are the following two definitions:

(Ne) The number that belongs to the concept *F* is the extension of the concept 'equinumerous to the concept *F*'.
(Nε) The number that belongs to the concept *G* is the extension of the concept 'equinumerous to the concept *G*'.

By (Nd), stating that the two extensions are identical, itself derived from (Na), we can infer our desired identity statement (Nb). What we have done is derive (Nb) not *directly* from (Na), but *indirectly* via (Nd) and the explicit definitions. So if we felt unhappy about Frege's explicit definition, but found the contextual method legitimate, we could still accept Frege's starting-point.[20]

It remains the case, then, that (Na) and (Nb) form the basis of Frege's logicist project:

(Na) The concept *F* is equinumerous to the concept *G*.
(Nb) The number of *F's* is identical with the number of *G's*.

Since (Nb) can be derived from (Na), with (or without) the help of the explicit definition (Ne), the task is now to provide a logical characterization of (Na). What is it for two concepts to be equinumerous? The answer has already been indicated: equinumerosity can be defined in terms of one-one correlation (*GL*, §63). Frege gives an example to illustrate the idea:

> If a waiter wants to be sure of laying just as many knives as plates on a table, he does not need to count either of them, if he simply lays a knife right next to each plate, so that every knife on the table is located right next to a plate. The plates and knives are thus correlated one-one, by means of the same spatial relationship. (§70.)

Generalizing from this example, then, two concepts *F* and *G* are equinumerous if there is a relation *R* that correlates one-one the objects falling under *F* with the objects falling under *G*. As we saw in §3.3, the logical analysis of one-one correlation had already been provided in the *Begriffsschrift*; so let us simply write down the condition as follows, '*Rxy*' symbolizing that *x* stands in relation *R* to *y*:

(Na*) $(\forall x)\,(Fx \rightarrow (\exists y)\,[\,Gy\ \&\ (\forall z)\,(Rxz \leftrightarrow z = y)\,]\,)$
$\&\ (\forall y)\,(Gy \rightarrow (\exists x)\,[\,Fx\ \&\ (\forall w)\,(Rwy \leftrightarrow w = x)\,]\,)$.

The first conjunct says that for any *F* (i.e. anything that is an *F*), there is one and only one *G* to which it is *R*-related, and the second conjunct adds that for any *G*, there is one and only one *F* to which it is *R*-related. (The first clause, in other words, states the condition for the relation between the *F's* and the *G's* to be *many-one*, and the second clause the condition for the relation to be *one-many*, the two clauses providing the combined condition for the relation to be *one-one* – compare (OO) in §3.3 above.) As Frege remarks in §72 of the *Grundlagen*, in offering the same analysis there, this 'reduces one-one correlation to purely logical relationships'.

Returning to the problem that Frege felt had been unresolved in §56, we do now have a way of determining whether the number that belongs to the concept *F* is the same as the number that belongs to the concept *G*. Frege's definitions of propositions of the form 'The number *n* belongs to the concept *F*' ('There are *n F's*') were regarded as unsatisfactory, because they did not adequately determine the relevant objects. But propositions of this form, according to Frege, are reducible to propositions that have the preferred form of an identity statement:

(NF) The number *n* is the number that belongs to the concept *F*.

The expression '*n* is a number' is taken as equivalent to the expression 'there exists a concept such that *n* is the number that belongs to it'; and this is now seen as acceptable with Frege's explicit definition (Ne) in place. 'Thus the concept of number is defined, admittedly, it seems, in terms of itself, but nevertheless without error, since "the number that belongs to the concept *F*" is already defined [as "the extension of the concept *equinumerous to the concept F*"].' (§72.)

All that is then needed, to provide definitions of the individual numbers, is to find appropriate concepts to substitute in (NF). In the case of the number 0, Frege utilizes the concept *not identical with itself*, yielding the following definition (cf. §74):

(N0) The number 0 is the number that belongs to the concept *not identical with itself*.

In offering this, Frege writes:

> Some may find it shocking that I should speak of a concept in this connexion. They will object, very likely, that it contains a contradiction and is reminiscent of our old friends the square circle and wooden iron. Now I believe that these old friends are not so black as they are painted. To be of any use is, I admit, the last thing we should expect of them; but at the same time, they cannot do any harm, if only we do not assume that there is anything which falls under them. That a concept contains a contradiction is not always obvious without investigation; but to investigate it we must first possess it and, in logic, treat it just like any other. All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp, that we should be able to decide definitely about every object whether it falls under that concept or not. But this demand is completely satisfied by concepts which, like "not identical with itself", contain a contradiction; for of every object we know that it does not fall under any such concept. (*FA*, §74.)

Furthermore, the crucial point about Frege's chosen concept is that it can be specified purely logically ('$x \neq x$'). As Frege remarks, 'I could have used for the definition of nought any other concept under which no object falls. But I have made a point of choosing one which can be proved to be such on purely logical grounds; and for this purpose "not identical with itself" is the most convenient that offers, taking for the definition of "identical" the one from Leibniz given above [§65], which is in purely logical terms' (ibid.). From (Ne) and (N0) we can then formulate an explicit definition that satisfies Frege's requirements:

(E0) The number 0 is the extension of the concept 'equinumerous to the concept *not identical with itself*'.

Assuming, with Frege, that the notion of an extension is unproblematically a logical notion, then we have indeed managed to characterize the number 0 in purely logical terms.[21]

The next step in the project is to define the successor relation, relating any two adjacent members of the natural number sequence. Frege offers this definition of '*n* follows in the series of natural numbers directly after *m*':

(SR) There is a concept *F*, and an object *x* falling under it, such that the number that belongs to the concept *F* is *n* and the number that belongs to the concept *falling under F but not identical with x* is *m*. (§76.)

Intuitively, this clearly gives the desired result: there is one less object falling under the latter concept than under the former, and the relationship between the two concepts can be characterized purely logically (cf. (F_{n+1}*) in §4.2 above).

Frege goes on to show how the definition yields 1 as the successor of 0 (§77). Take the concept *identical with 0*. Since one and only one object falls under this concept, namely, the number 0, the number that belongs to this concept is the number 1. The number that belongs to the concept *falling under the concept 'identical with 0' but not identical with 0*, on the other hand, is clearly 0, since nothing can fall under this concept. So the condition stated in (SR) is satisfied (taking '*F*' as 'identical with 0', giving $x = 0$, $n = 1$ and $m = 0$), and we can conclude that 1 is the successor of 0. What Frege has done here, in other words, is provide a suitable concept to substitute in (NF) to generate a definition of the number 1:

(N1) The number 1 is the number that belongs to the concept *identical with 0*.

Since 0 has already been defined purely logically, and in fact is the only object that has been so defined up to this point, the concept *identical with 0* is obviously the ideal concept for Frege to take to logically define the number 1. What the argument just given then shows is that this is indeed the number that follows in the series of natural numbers directly after 0.

From (Ne) and (N1), the following explicit definition can then be offered:

(E1) The number 1 is the extension of the concept 'equinumerous to the concept *identical with 0*'.

With the numbers 0 and 1 now defined, the number 2 can then be generated in a similar way:

(N2) The number 2 is the number that belongs to the concept *identical with 0 or 1*.

(E2) The number 2 is the extension of the concept 'equinumerous to the concept *identical with 0 or 1*'.

The pattern that emerges is clear: each number can be defined in terms of its predecessor(s), since the series of natural numbers up to a given number *n* has itself $n + 1$ members (since it starts from 0). This suggests the following general definition:

(Nn+1) The number $n + 1$ is the number that belongs to the concept *member of the series of natural numbers ending with n*.

Of course, the concept *member of the series of natural numbers ending with n* itself needs to be defined, but once again, the materials for doing so had already been supplied in the *Begriffsschrift*. As we saw in §3.3, a logical characterization had been offered, through the notion of an hereditary property, of '*b* follows *a* in the *f*-series' (cf. *GL*, §79), from which '*b* is a member of the *f*-series beginning with *a*' could then be defined (cf. *GL*,

§81). Since this is equivalent to '*a* is a member of the *f*-series ending with *b*', the required logical definition can be provided. (SR) can then be used to show that (Nn+1) yields $n + 1$ as the successor of n – substituting 'member of the series of natural numbers ending with *n*' for '*F*', '*n*' for '*x*', '$n + 1$' for '*n*', and '*n*' for '*m*'.[22]

With Frege's definitions in place, it becomes possible to derive the familiar properties of the natural numbers. For example, (Nn+1) implies that every number has a successor, i.e. that no member of the series of natural numbers follows after itself, as Frege puts it in §83. In the *Grundlagen* Frege merely states a handful of theorems; the full task was to be undertaken in the *Grundgesetze*.[23] But if we grant him his assumptions, we can understand the optimism with which he faced this task. Certainly, his apparent success in generating the entire natural number series from the (purportedly logical) definitions offered would have made the logicist project look thoroughly feasible. As Frege himself concluded in Part V of the *Grundlagen*, 'I hope in this work to have made it probable that arithmetical laws are analytic judgements and therefore *a priori*. Accordingly, arithmetic would be simply a further developed logic, every arithmetical theorem a logical law, albeit a derivative one.' (§87.)

4.4 Non-natural Numbers

Most of the positive part of Frege's *Grundlagen* is concerned with showing how the natural numbers can be defined logically. Except for some brief remarks about infinite (transfinite) numbers (§§84-6), it is only in Part V, entitled 'Conclusion', that Frege addresses the question of other numbers, and even here the main thrust of the discussion seems to be to refute formalism. The formalist is understood as someone who imagines that one need only postulate that, say, the laws of addition and multiplication, as defined over the natural numbers, hold for any extension of the number system, in order to coherently investigate the properties of that extended system.[24] But, Frege argues, it is quite wrong to suppose that a concept has instances if no contradiction has yet revealed itself – not only are self-contradictory concepts admissible, but even if a concept contains no contradiction, that is still no guarantee that anything falls under it.[25] So more is needed in justifying the introduction of new numbers than the formalist would seem to imply.

Frege remarks that 'It is common to act as if mere postulation were already its own fulfilment' (§102), a criticism that later found expression in Russell's famous comment that 'The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil' (*IMP*, p. 71). The real work of proving consistency cannot be shirked. 'Postulating', say, that through any three points a straight line can be drawn is simply incoherent; and as we noted in §3.1, quaternions provide just one example of numbers which fail to obey what had previously been regarded as a fundamental law, in this case concerning the commutativity of multiplication. (Cf. §102.) With the introduction of new numbers, Frege remarks, 'the meanings of the words "sum" and "product" are extended' (§100), and we cannot automatically assume that initial definitions of basic concepts remain valid in any enlarged system.[26]

But if we cannot just define new numbers into existence by specifying a list of properties that characterize them, nor arrive at them by simply extending an existing number system taking its axioms for granted, how are they then to be apprehended? In asking this question in §104, Frege answers that 'everything will depend in the end on finding a judgeable content that can be transformed into an equation [identity] whose sides are precisely the new numbers. In other words, we must fix the sense of a recognition judgement [*Wiedererkennungsurtheils*] for such numbers.' Just as it did in the project of defining the natural numbers, what underlies Frege's strategy is his adoption of the context principle, and it is worth noting here that at the very point at which Frege introduced the principle, in §60, he took the example of infinitesimals to illustrate its importance. As we saw in §3.1, if infinitesimals are understood as actually existing quantities, then a contradiction ensues, and the task seemed to be to find some way of construing them as 'useful fictions', as Leibniz had regarded them. In this respect, Frege's approach looks promising: 'It all depends on defining the sense of an equation [identity] of the form $df(x) = g(x)dx$, rather than showing that there is a line bounded by two distinct points whose length is dx' (§60, n.).

However, this only raises the question as to whether contextually defined terms nevertheless do refer to genuinely existing objects, an issue on which Frege's and Leibniz's views seem to diverge.[27] Certainly, Frege himself, in reminding us in §104 of his general strategy of 'splitting up a content' to yield an identity, immediately goes on: 'In doing so, we must heed the doubts that we discussed, in §§63-8, concerning such a transformation. If we proceed in the same way as we did there, then the new numbers will be given to us as extensions of concepts.' And these extensions, of course, are themselves regarded as objects, albeit logical ones. But since Frege says no more about infinitesimals in the *Grundlagen*, and makes only a few remarks in his later work, it is hard to be clear as to his precise position.[28]

Frege does, however, express firmer views on those numbers at the opposite end of the spectrum, namely, transfinite numbers, which he briefly discusses in the final subdivision of Part IV, immediately after his sketch of the logicist reduction of the natural numbers. The idea is simple. Consider the concept *finite number*, defined as the concept *member of the series of natural numbers beginning with 0* (cf. §83), which, as we saw in the last section, can be logically characterized and applies to all the natural numbers. The number that belongs to this concept is clearly an infinite number, and Frege symbolizes it by '∞_1'. He writes:

There is nothing at all weird or wonderful about the infinite number ∞_1 so defined. 'The number that belongs to the concept F is ∞_1' means [*heisst*] no more nor less than: there is a relation that correlates one-one the objects falling under the concept F with the finite numbers. According to our definitions, this has a perfectly clear and unambiguous sense; and that is sufficient to justify the use of the symbol ∞_1 and secure it a meaning [*Bedeutung*]. That we can form no idea of an infinite number is quite irrelevant and applies just as much to finite numbers. Our number ∞_1 is in this way just as definite as any finite number: it can without doubt be recognized as the same again and be distinguished from another. (§84.)

Frege goes on to express his agreement with Cantor that infinite numbers are as legitimate as finite numbers (§85), though he does suggest that his own method of introducing infinite numbers, through logical definition, is superior to Cantor's appeal to 'inner intuition' (§86). Furthermore, Frege notes, since numbers are characterized right from the start as belonging to concepts, there is no extension of the meaning of 'number' when infinite numbers are introduced (since they too are attached to concepts), so that worries about invalidating any fundamental laws are minimized (§85).

The case of infinite numbers suggests that if even they – amongst the most controversial of numbers – can be defined on the basis of the natural numbers, then so too can all the other numbers, and although Frege hardly mentions the real numbers in the *Grundlagen*, there is no indication that he did not regard them as similarly definable.[29] But it is only in the second volume of the *Grundgesetze*, published in 1903, that Frege finally takes up the issue; and then it comes as a surprise to learn that he believes that the natural numbers and the real numbers form quite distinct domains. This constitutes a departure from what, by the end of the 19th century, as we indicated in §3.1, had become the dominant foundationalist tradition. Frege is perfectly aware of this, and in fact introduces his own positive account of the real numbers, much as he did in the *Grundlagen* with regard to the natural numbers, with a lengthy and scathing attack on rival conceptions.[30]

Frege's main criticism of these alternative views concerns their failure to accord due weight to the *application* of real numbers. Defining real numbers in terms of converging sequences of rational numbers, for example, ignores their central role in *measurement*. Frege writes:

it is not possible to extend the domain of natural numbers to that of the real numbers; they are quite separate domains. The natural numbers answer the question 'How many objects of a certain kind are there?', whilst the real numbers can be regarded as measurement numbers [*Maasszahlen*], which state how large a magnitude is compared with a unit magnitude. (*GG*, II, §157.)[31]

According to Frege, a real number is essentially a *ratio of magnitudes* (*Grössenverhältnis*); and this precisely captures its role in measurement (cf. *GG*, II, §73). Such an emphasis on the *application* of numbers might seem in conflict with the attitude he revealed in the *Grundlagen* in accusing Mill of confusing the applications of an arithmetical proposition with the pure proposition itself (*GL*, §9; see §3.4 above). But Frege was quite insistent even then that a proper account of a number system should *explain* its applicability. In the case of the natural numbers, for example, it was construing them as belonging to concepts, according to Frege, that explained their own wide-ranging applicability – to both physical and mental phenomena (cf. *GL*, §48; see §4.1 above). Whilst *particular applications* of a number system are distinct from its essential functioning, its *general applicability* is not: the features responsible for those applications are fundamental.[32]

Frege remarks that 'exactly the same ratio of magnitudes, which obtains between line segments, also obtains between intervals of time, masses, intensities of light, and so on. In this way the real number is detached from these particular kinds of magnitude and is, as it were, suspended above them.' (*GG*, II, §158.) He goes on to distinguish his own position both from the old geometrical conception, which, whilst misguidedly based on spatial intuition, at least treated real numbers as measurement numbers, and from more recent accounts, in which 'measurement either fails to appear at all or else is patched on purely externally, without an inner connection grounded in the essence of number' (*GG*, II, §159). So whilst real numbers are independent of particular magnitudes, they are far from independent of magnitude in general.

Just as whatever falls under a sortal concept can be counted, then, so too whatever can be measured can be assigned a real number; and a theory of the real numbers must explain the latter just as much as a theory of the natural numbers must explain the former. Despite the difference in the roles of these two types of number, however, there are similarities in Frege's accounts of them. For just as the natural numbers are identified with extensions of logically definable concepts (functions of one argument), so too the real numbers are identified with extensions of logically definable relations (functions of two arguments).

The key notion in Frege's account of the real numbers is that of a *magnitude domain* (*Grössengebiet*).[33] A magnitude domain is any class of objects with the type of structure revealed, for example, in the domains of lengths, masses, or temperatures (cf. *GG*, II, §161). Any two objects within such a domain stand in a certain relation to one another, i.e. a ratio of magnitudes obtains; and it is the extensions of these relations that exhibit the features of the real numbers. The task is then to find an appropriate class, which Frege terms a *positive class*, possessing the required properties of a magnitude domain from which the real numbers can be defined.

The objects of the chosen class must themselves, of course, be logically definable. To indicate the feasibility of his project, Frege notes that every real number can be represented in the following form, where 'r' is understood as a non-negative integer, and 'n_1', 'n_2', etc. constitute an infinite sequence of positive integers (*GG*, II, §164):

$$r + \sum_{k=1}^{k=\infty} \frac{1}{2^{n_k}}$$

Every real number can thus be represented as an ordered pair, consisting of a natural number and a class of natural numbers excluding 0. Since the natural numbers can be defined logically, on Frege's view, so too can the ordered pairs and hence the relations between them. The extensions of these relations can then be taken as the objects of a positive class, the extensions of the relations between these objects defining the real numbers. The real numbers thus end up as the extensions of relations between extensions of relations between ordered pairs of extensions of concepts and sets of extensions of concepts. What could be more logical?

After indicating the outline of his project (*GG*, II, §§156-64), Frege spends the rest of Volume II of the *Grundgesetze* formally demonstrating the properties of a positive class. In §245 he notes his next task – to show that there exists a positive class; but with this note his constructive account ends. A third volume was clearly planned, but by this time, Frege had been informed by Russell of the paradox that effectively dealt a death-blow to his logicist ambitions.[34] Despite an appendix to the second volume, added whilst the book was in press, that sought to respond to the paradox, Frege was never able to complete the project begun so optimistically with the publication of the *Begriffsschrift* and single-mindedly pursued for over two decades with such technical and philosophical brilliance.

4.5 Unnatural Numbers?

Nine years passed between the publication of Frege's *Grundlagen* in 1884 and the appearance of Volume I of the *Grundgesetze* in 1893, and a further ten years before Volume II saw the light of day in 1903; and most of Frege's energies during this period were indeed taken up with the detailed working out of his logicist programme.[35] Volume I of the *Grundgesetze* contains 179 sections, divided into two Parts, Part I explaining his 'Begriffsschrift' – his symbolism, axioms and definitions (§§1-52), and Part II presenting formal proofs of the fundamental laws of arithmetic (§§53-179). Part I constitutes a revised and richer elaboration of the logical system introduced in the *Begriffsschrift*; and Part II demonstrates rigorously what was merely sketched in the constructive part of the *Grundlagen*. Part II, in fact, continues into Volume II of the *Grundgesetze* (§§1-54); and Part III of the *Grundgesetze*, on the real numbers, begins at §55 of Volume II. As

already remarked, Frege's account of the real numbers follows the pattern of his account of the natural numbers: a critique of rival views precedes the formal presentation of his own theory.[36]

Two fundamental developments in Frege's thought between the *Grundlagen* and the *Grundgesetze* may be mentioned here, both of which are discussed in detail in the chapters that follow. Firstly, Frege introduced his distinction between *Sinn* and *Bedeutung*, and secondly, connected with this, he clarified his ontology, sharpening his views about objects and concepts. Most notably, *truth-values*, themselves admitted as objects, were taken as the *Bedeutungen* of sentences, so that concepts could be construed as functions mapping objects onto truth-values (see §6.2 below); and *extensions of concepts* too became accepted as legitimate objects, and hence as possible arguments of functions. Although Frege had introduced extensions of concepts in the *Grundlagen*, he had not at the time felt committed to their use. In a footnote to §68, for example, where he first offers his definition of number, he remarks: 'I believe that for "extension of the concept", simply "concept" could be said.' He raises two objections to this, firstly, that numbers (unlike concepts) are objects, and secondly, that concepts can have the same extension without themselves coinciding, but then immediately states without argument that both objections can be met. In reviewing his enquiry at the end of the book, he repeats that 'I attach no great importance to the introduction of extensions of concepts' (§107; see §5.5 below). In fact, far from being met, the two objections were soon sustained, as Frege thought through his own views more carefully (see §7.1 below). The result was that the definitions of the *Grundlagen* remained firmly in place. What happened between the *Grundlagen* and the *Grundgesetze*, then, can be seen as reinforcing, and indeed as offering justification for, Frege's appeal to extensions of concepts.

Frege's considered logicist position was thus that the natural numbers can be defined as extensions of concepts, and the real numbers as extensions of relations. I shall examine Frege's argument for such identifications, as originally given in the *Grundlagen*, in more detail in §5.3, and consider other aspects of his logicism in subsequent sections. But in the final section of this chapter, I want to comment on the plausibility of this account. For the obvious criticism of it is that whilst Frege may, at best, have succeeded in defining objects that have all the key properties of the relevant numbers, he has not defined the natural or the real numbers themselves. Is it not *unnatural*, we might ask, to construe the former as extensions of concepts, and *unreal* to construe the latter as extensions of relations between extensions of relations between ordered pairs of extensions of concepts and sets of extensions of concepts?

Taking the case of real numbers first, let us approach the problem by briefly comparing Frege's conception with the theories of Cantor and Dedekind mentioned in §3.1. The main difference is that whilst Cantor and Dedekind seek to define the real numbers in terms of sequences or classes of rational numbers, Frege attempts to define them using only the natural numbers, albeit in a more complicated fashion.[37] Given that the rationals can themselves be defined as ratios of natural numbers, the difference might seem of little significance. But as we saw in the last section, one of Frege's fundamental concerns was to provide an explanation of the *application* of real numbers, and according to him, Cantor's and Dedekind's theories, whatever their straightforward appeal might be, fail to do justice to the role of the real numbers as measurement numbers. Frege's 'unreal' account at least accords a central place to the idea of a 'ratio of magnitudes'.

But there is a deeper issue involved here, which surfaces in Frege's critique of Cantor's theory, in particular (*GG*, II, §§68-85). Cantor writes that he 'assigns' what he calls a 'fundamental sequence' (i.e. a convergent series of rationals) to the number *b* to be defined, and Frege opens his critique by quoting this remark.[38] But if '*b*' simply *designates* the fundamental sequence, Frege argues, then we have not yet reached the real numbers themselves; if, on the other hand, we take it that '*b*' does indeed designate a number, then are we not presupposing what it is our very task to define? The dilemma is clear: either the real numbers are *identified* with fundamental sequences, which does not, in fact, individuate them as numbers; or else they are merely *correlated* with fundamental sequences, which assumes that they are already given, a fundamental sequence simply determining *which number* '*b*' is to stand for. In neither case are the real numbers adequately defined.

The obvious answer to this dilemma is to allow that we do have antecedent knowledge of what it is for a number to be the limit of a fundamental sequence – in the case of rationals. 1, for example, can be represented as the limit of the series 1/2, 2/3, 3/4, ... $\upsilon/(\upsilon+1)$, ... (cf. *GG*, II, §84); and in general, any rational number *a* can be represented, trivially, as the limit of the sequence all of whose terms are *a* (i.e. *a*, *a*, *a*, ...). The idea is then simple: wherever a fundamental sequence does *not* have a rational limit, *there* we have an irrational number. As Frege himself puts it, interpreting Cantor as favourably as he can, 'With every fundamental sequence there is connected a certain number, which does not need to be rational. These numbers are therefore partly new, not as yet considered, and they are to be determined precisely by the fundamental sequence with which they are connected.' (*GG*, II, §77.) Starting with the rationals, then, it would seem that we can indeed define a new set of numbers, simply adding to our existing system.

According to Frege, however, this does not avoid the underlying dilemma. For either the real numbers are assumed as given, or they are not. If they are, and we merely have a correlation established between them and fundamental sequences, then definitions are still lacking. If they are not, then this holds as much for the rational as for the irrational real numbers. There is nothing in Cantor's account to justify *identifying* the

real number defined as the limit of a fundamental sequence such as *a*, *a*, *a*, ... with the *rational number a* itself. If such an identification seems obvious, then that is only because we are presupposing an understanding of a number being the limit of a series, which is precisely what requires explanation.[39]

These considerations confirm Frege in his view that the real numbers must be defined as a complete system, and not piecemeal via the rational numbers. Of course, as we have seen, Frege held this view for an independent reason, namely, that both the rational and irrational numbers are measurement numbers, and must be treated together. But if we play down the requirement that measurement be made central in any account (allowing it to be 'patched on externally'), whilst accepting the point that the real numbers form a distinct system from the rationals, then it is possible to refine Cantor's theory along otherwise Fregean lines. For Frege himself admits that Cantor adequately defines what it is for two fundamental sequences to have the same limit: two sequences (a_υ) and (a'_υ) have the same limit iff $(a_\upsilon - a'_\upsilon)$ vanishes (i.e. tends to 0) as υ tends to infinity (cf. *GG*, II, §81). The real numbers can then be defined as *equivalence classes* of fundamental sequences using just that method of logical abstraction that Frege introduced in the *Grundlagen*.[40]

Dedekind's theory, to which similar objections can be raised, can also be refined along Fregean lines, a real number in this case being identified, say, with the lower of the two classes determined by a Dedekind cut.[41] If these refinements are made, then aside from the issue of the application of numbers, we would seem to have a variety of equally legitimate, i.e. logically acceptable, ways of introducing the real numbers, each of which identifies any given real number with a *different* class. But if this is so, then the idea of a *uniquely correct* account of the real numbers disintegrates. Indeed, it would no longer seem appropriate to talk of *the* real numbers at all; what count as the real numbers would depend on the underlying theory. As long as the objects defined – whatever classes they might be identified with – have all the right properties, then an adequate characterization is provided.

The possibility of alternative construals is clearly incompatible with Frege's conviction that each number is an individual object, as indicated, according to Frege, by the fact that we do speak of *the* number one, for example. Of course, we might allow that there can be alternative construals of the real numbers, but insist that the natural numbers, in terms of which the real numbers are defined, are unique. But since Frege's work, the development of set theory has generated a variety of reductions of the natural numbers too. Most accounts start by defining 0 as the null set, and 1 as the unit set of the null set (i.e. the set that contains the null set as its sole member); but thereafter more than one way of proceeding is possible. Zermelo (1908), for example, defined 2 as the unit set of 1, 3 as the unit set of 2, and so on (each number, except 0, being the unit set of its predecessor); whilst von Neumann (1923) defined 2 as the set composed of 0 and 1, 3 as the set composed of 0, 1 and 2, and so on (each number, except 0, being the set of its predecessors). Symbolizing the null set by '∅', the following two series, respectively, are thus generated:

(ZS) $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$
(NS) $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$

The underlying set theories will themselves have different properties and applications, but each is capable of generating a series with which the natural numbers can be identified.[42]

Frege himself would have seen these possibilities as only lending support to his own theory: what they show is that 2, for example, should be identified not with any *particular class* such as represented in (ZS) or (NS), but with the *class of all such classes* (i.e. with the class of all classes equinumerous to a given 2-membered class). Of course, this class too is a particular class, but since it comprises all 2-membered classes, and can be characterized purely logically, it accords best of all with Frege's fundamental assumption that logic is the most general of sciences, governing everything that is thinkable.[43] Unfortunately, however, Russell's paradox was to reveal the dangers involved in so naively talking of classes of classes, so that the modern set theorist is forced to abandon Frege's own definitions and follow a path such as taken by Zermelo or von Neumann.[44]

In his influential paper, 'What numbers could not be' (1965), Paul Benacerraf argues that the possibility of alternative set-theoretic reductions of the natural numbers shows that numbers *cannot* be sets. If numbers *were* sets, then it must be possible to say *which ones* they are; but a unique specification is what cannot be given. Benacerraf further argues that numbers cannot be objects at all, but this conclusion does not itself follow. What is shown, at best, is that numbers cannot be identified with objects of any other kind, such as sets, numerals, directions, or human beings, but not that they are not objects at all. Of course, as we saw in §4.2, it was the Caesar problem that led Frege to identify numbers with extensions of concepts, the criteria of identity for which we were assumed to know; but if we permit the introduction of abstract objects at all, through some kind of contextual definition, and Frege allows extensions of concepts themselves, then there is no reason, on this basis at least, why numbers should not be objects in their own right.[45]

However, there remains a problem about Frege's identification of numbers with extensions of concepts. For even if his definitions give logicism its greatest chance of success, numbers do not immediately strike anyone as extensions of concepts, as Frege himself admits.[46] It is at this point that Frege is forced to become more philosophical in his argument. As we shall see in §5.5, Frege ends up emphasizing that he is *reconstructing* the natural numbers. Prior to the rigorous development of a theory, Frege argues, the question as to which objects *the* natural numbers are has no

clear sense, and it is precisely the task of the theorist to provide it with one. That numbers do not appear to us, pre-theoretically, to be extensions of concepts is not, therefore, an objection. Of course, this response generates further problems of its own, not least of which concerns Frege's notion of *sense*; and I discuss this notion in detail in chapter 6.

Here we need only note that despite Frege's later attempts at justification, there remains a crucial ambiguity in his conception of his reconstructive project. For does he regard himself as uncovering for the first time the real essence of those numbers with which we have all along been familiar, or as setting up a new system in which we have *analogues* of the natural numbers? If only the latter, can we truly say that Frege has shown that *the* natural numbers are logical objects? Although officially, Frege was undoubtedly a *realist* – mathematical objects are indeed already there waiting to be discovered and described, there are nevertheless aspects of his mathematical and semantic work that are in conflict with such a position; and I return to this issue in the final two chapters.

First, however, we must elucidate the underlying dilemma, which, in relation to Frege's logicist project, may be stated as follows. If we are to assess Frege's project, then we must clearly have a grasp of the basic properties of numbers (properties which Frege himself refers to as 'well-known'; *GL*, §70). But if we do, must we not *already know* whether numbers are extensions of concepts or not? Obviously, if they are not, then Frege's definitions are simply incorrect. But if they are, how is it that we seem not to know such a fundamental fact? What surfaces here is the so-called *paradox of analysis*, and it is the task of the next chapter to explain the general problem and the development of Frege's philosophy in response to it.