APPENDIX 2

Leibniz on Analysis

All proof, according to Leibniz, depends on the Principle of Identity, which states that a proposition expresses an 'identity' if the predicate is explicitly either identical with or included in the subject. Now Leibniz believed that 'always, in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject' (*LAC*, p. 63); so that if this inclusion could be made explicit, if it was not already, a *proof* of the proposition could be achieved. Proof thus proceeds by reducing the proposition to an 'identity' by successive applications of the rule of 'substitution of equivalents', i.e. by *analysing* each term in turn by means of a definition. As illustrations, consider the following two proofs of '2 + 1 = 3' ('Two plus one is three') and 'Logicians are thinkers', respectively:

(A)	(1)	2 + 1	=	3	
	(2)	1 + 1 + 1	=	3	(by the def. '2 = 1 + 1')
	(3)	1 + 1 + 1	=	2 + 1	(by the def. '3 = 2 + 1')
	(4)	1 + 1 + 1	=	1 + 1 + 1	(by the def. '2 = 1 + 1')

(B) (1) Logicians are thinkers

(2) Sharp-minded philosophers are thinkers
(by the def. 'Logicians are sharp-minded philosophers')

(3) Sharp-minded enlightened intellectuals are thinkers
(by the def. 'Philosophers are enlightened intellectuals')

(4) Sharp-minded enlightened stimulating thinkers are thinkers
(by the def. 'Intellectuals are stimulating thinkers')

In both cases, (4) is an 'identity' in Leibniz's sense, and cannot itself be demonstrated, being something that is immediately *intuited* as true. (Cf. MacDonald Ross, 1984: pp. 62-3; Ishiguro, 1990: pp. 56-7.) But if it is possible to *analyse* downwards from (1) to (4), to *prove* a proposition, then it must be possible to *synthesize* upwards from (4) to (1), to *discover* (the fact expressed by) a proposition. (Cf. Kneale, 1962: pp. 332-3.) Once the basic terms are found, Leibniz believed, we can move in *either* direction, and a universal character can thus be combined with a *calculus ratiocinator* to provide *both* a logic of proof *and* a logic of discovery (cf. §2.1 above).

APPENDIX 3

Frege's Logical Notation

Frege called his logical notation 'Begriffsschrift', which literally means 'concept-script' (it has also been translated as 'conceptual notation'), reflecting his avowed aim of providing a means of capturing the 'conceptual content' ('Begriffsinhalt') of propositions (see §2.5 above). The name also formed the title of his first book, which introduced his logical system. Since I have used modern notation in discussing Frege's ideas in the present work, a brief account of Frege's own symbolism and its translation into modern notation is provided here.

Frege explains his symbolism in Part I of the *Begriffsschrift*. He starts by introducing the following symbol (§2):

⊢——

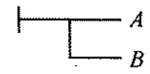
This is seen as made up of a *horizontal stroke*, which Frege calls the *content stroke* (*Inhaltsstrich*), which indicates that what follows is a 'content' that can be asserted (i.e. is the 'content' of a proposition that can be judged to be true), and a *vertical stroke*, which Frege calls the *judgement stroke* (*Urtheilsstrich*), which indicates that the content is indeed asserted (i.e. that the relevant proposition is true). The judgement that the proposition *A* is true is thus represented as follows:

⊢—— *A*

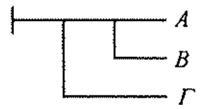
In §5 Frege notes that if *A* and *B* represent judgeable contents (i.e. are propositions), then there are four possibilities to consider (anticipating Wittgenstein's introduction of *truth-tables* in the *Tractatus*, a treatise that was profoundly influenced by Frege's work):

(i) *A* is affirmed, *B* is affirmed (i.e. both are true);
(ii) *A* is affirmed, *B* is denied (i.e. *A* is true, *B* is false);
(iii) *A* is denied, *B* is affirmed (i.e. *A* is false, *B* is true);
(iv) *A* is denied, *B* is denied (i.e. both are false).

The following symbol is then defined as representing the judgement that *the third of these possibilities does not obtain, but one of the other three does*:

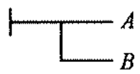
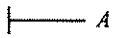


What is represented here is the assertion of the conditional proposition 'If *B*, then *A*', understood as involving the material conditional, i.e. construed as what would be formalized in modern notation as '$B \rightarrow A$', since this is indeed equivalent to '$\neg(\neg A \ \& \ B)$' (denying the third possibility). The vertical stroke that connects the upper and lower horizontal strokes Frege thus calls the *conditional stroke* (*Bedingungsstrich*). More complex symbols can then be readily constructed, such as the following:



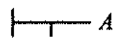
This represents the assertion of the complex conditional proposition '$\Gamma \rightarrow (B \rightarrow A)$', which is equivalent to '$\neg(\neg A \ \& \ B \ \& \ \Gamma)$'. (Cf. *BS*, §5.)

Frege's definition of the conditional thus implies that from the judgements

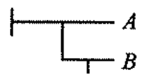
 and  the judgement  follows;

and what is involved here is *modus ponens*, the rule that licenses inferring '*A*' from '*B*' and 'If *B*, then *A*'. 'In logic', Frege writes, 'following Aristotle, a whole series of modes of inference are enumerated [cf. App. 1 above]; I use just this one – at least in all cases where a new judgement is derived from more than one single judgement'. If it is possible to manage with a single mode of inference, Frege goes on, 'perspicuity demands that this be done'. (*BS*, §6.)

Frege then defines the *negation stroke* (§7), which he represents by attaching a small vertical stroke to the underside of the content-stroke:



This is understood as *denying* (the content of the proposition) *A*, i.e. as asserting that '$\neg A$' is true. Using the conditional and negation strokes, further judgements can then be represented. For example, assertion of the proposition '*A* or *B*', construing 'or' in the inclusive sense, i.e. '$A \vee B$', which is equivalent to '$\neg B \rightarrow A$', can be represented as follows:



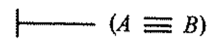
'$A \ \& \ B$', which is equivalent to '$\neg(B \rightarrow \neg A)$', can also be readily represented (note how the *scope* of each negation sign is reflected in Frege's symbolism):

What Frege has thus specified is a system of propositional logic, with negation and the conditional as the primitive connectives, and *modus ponens* as the basic rule of inference. (Frege also makes tacit use of a rule of substitution.) Both conjunction and disjunction can clearly be defined in Frege's system (as Frege himself explains in §7); but it has to be said that, except in the case of conditional propositions, Frege's two-dimensional 'Begriffsschrift' does not render the validity of inferences in propositional logic as perspicuous as modern notation is capable of doing. Of course, it is true that any system that uses some but not all of the propositional connectives either 'deflates' or 'inflates' certain inferences. As an example of each, consider the following 'translations' of one of de Morgan's laws, '$\neg(A \ \& \ B) \leftrightarrow \neg A \vee \neg B$', and one of the distributive laws, '$A \vee (B \ \& \ C) \leftrightarrow (A \vee B) \ \& \ (A \vee C)$', respectively:

(ML) '$(B \rightarrow \neg A) \leftrightarrow (B \rightarrow \neg A)$';
(DL) '$[(C \rightarrow \neg B) \rightarrow A] \leftrightarrow \neg[(\neg C \rightarrow A) \rightarrow \neg(\neg B \rightarrow A)]$'.

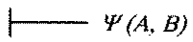
But Frege's notation would certainly add to the 'inflation' of many inferences; and although the possibility of defining '&' and '∨' in terms of '→' and '¬' is instructive (Frege himself frequently emphasizes the value of making do with as few primitives as possible), the lack of simple signs for conjunction and disjunction (even if defined in terms of other connectives) must be regarded as a deficiency of the symbolism.

In §8 Frege defines a symbol for what he calls 'identity of content' ('Inhaltsgleichheit'):



This is understood as meaning that 'the symbol *A* and the symbol *B* have the same conceptual content, so that *A* can always be replaced by *B* and vice versa'. As argued in §2.5 above, Frege's criterion for sameness of conceptual content is essentially *logical equivalence*; but the problems that this raises, and the restrictions that are needed in appealing to intersubstitutability *salva veritate*, were only recognized by Frege later, in his paper 'Über Sinn und Bedeutung' (see ch. 6 above). At the time of the *Begriffsschrift*, as §8 reveals, Frege also thought that identity of content was a relation between names rather than contents, a view which he criticizes at the beginning of 'Über Sinn und Bedeutung' (see §6.1 above).

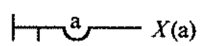
In the rest of Part I of the *Begriffsschrift* (§§9-12) Frege introduces his notation for what we now know as predicate logic. In §9 he explains his use of function-argument analysis (for details, see §5.2 above), and in §10 he defines the following symbols as '*A* has the property *Φ*' ('*Fa*' in modern notation) and '*B* stands in the *Ψ*-relation to *A*' ('*Rba*' in modern notation), respectively:



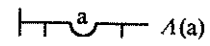
In §11 he introduces his key symbol for the universal quantifier, which involves inserting in the content stroke a concavity (*Höhlung*) in which the letter indicating the argument is placed:

This is understood as representing the judgement that 'the function [*Φ*] yields a fact whatever is taken as its argument', i.e. that everything has the property *Φ* (for all *x*, *Fx* – '(∀*x*) *Fx*' as it would be formalized in modern notation). [Frege uses italicized upper case Greek letters for function terms, lower case Gothic (old German) letters for bound variables, and italicized ordinary letters for free variables. I have replaced Frege's Gothic letters with ordinary letters in the account offered here. In modern notation, italicized upper case ordinary letters – such as *F* and *G* – are conventionally used for function terms, italicized lower case ordinary letters typically from the end of the alphabet – such as *x* and *y* – for bound variables, and italicized lower case ordinary letters typically from the beginning of the alphabet – such as *a* and *b* – for free variables, as illustrated in the inference schema '(∀*x*) *Fx* → *Fa*', reflecting the rule of universal elimination.] As Frege points out, the concavity with the letter written in it 'delimits the scope [*Gebiet*] of the generality signified by the letter' (§11). As shown in §2.2 above, appreciation of the *scope* of the quantifier is essential in formalizing statements of multiple generality, where ambiguity may be present.

In §12, Frege considers certain combinations of symbols. Using negation and the universal quantifier, existential judgements such as 'There are some things that do not have the property *X*' (understood as equivalent to 'It is not true that everything is *X*') can be represented:

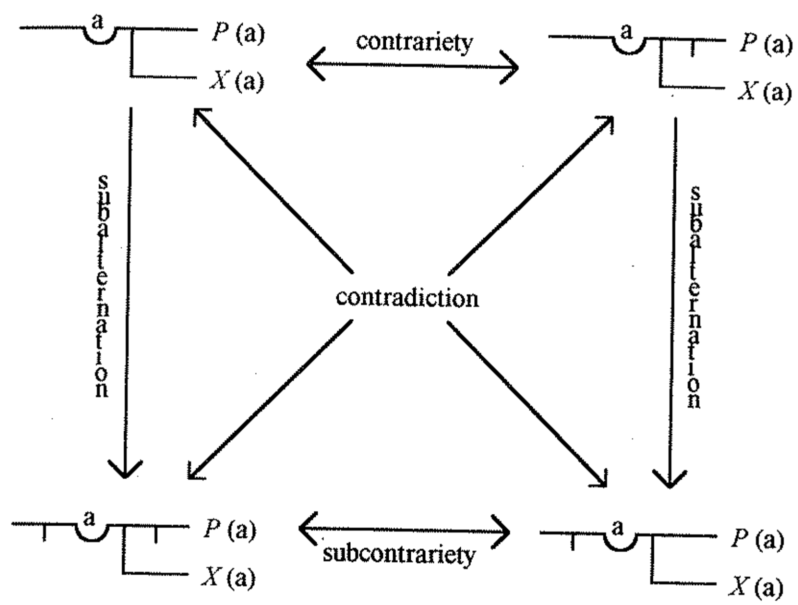


The following symbol, on the other hand, is translated simply as 'There are *Λ*'s' (i.e. 'There is at least one *Λ*'):



Clearly, what we have here is a definition of the existential quantifier – '(∃*x*) *Fx*', in modern notation, being definable as '¬(∀*x*)¬*Fx*' – although as in the case of conjunction and disjunction, Frege does not introduce a simple sign for the existential quantifier.

Frege also shows how the four types of syllogistic proposition, as contained in the traditional square of opposition (cf. App. 1 above), are represented in his system:



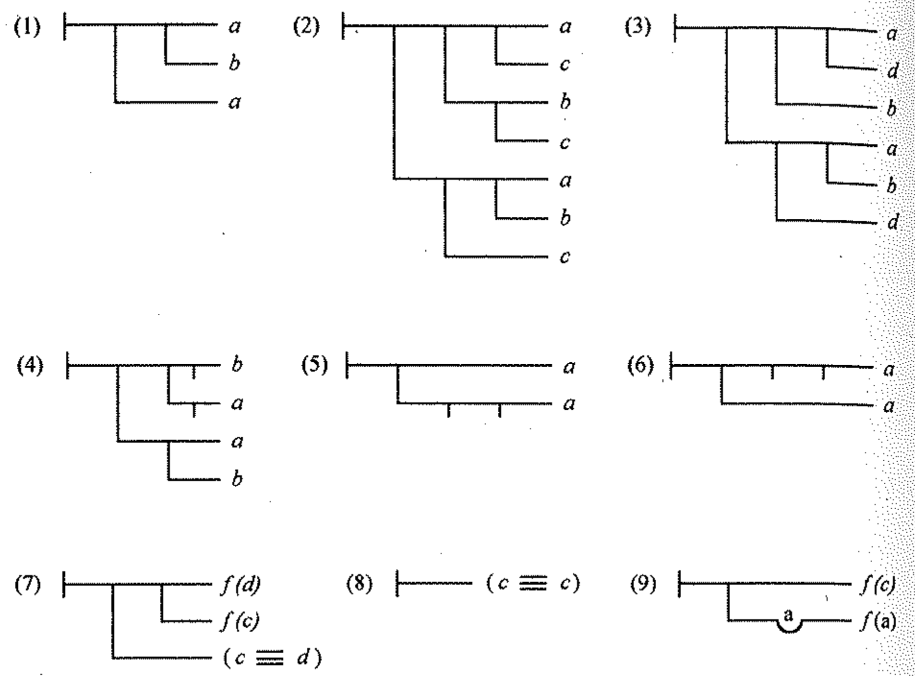
Syllogistic A, E, I and O propositions are thus formalized, in modern notation (substituting '*C*' for '*X*' and '*x*' for 'a'), as follows:

(A) (∀*x*) (*Cx* → *Px*);
(E) (∀*x*) (*Cx* → ¬*Px*);
(I) ¬(∀*x*) (*Cx* → ¬*Px*), which is equivalent to (∃*x*) (*Cx* & *Px*);
(O) ¬(∀*x*) (*Cx* → *Px*), which is equivalent to (∃*x*) (*Cx* & ¬*Px*).

As noted in Appendix 1, however, although the square of opposition is reproduced in §12 of the *Begriffsschrift*, with the traditional relations marked, Frege does not make clear that contrariety, subcontrariety and subalternation are all invalidated under his formalizations. [(A) and (E) can both be true (if there are no *C*'s); (I) and (O) can both be false (if there are no *C*'s); and (A) does not imply (I), and (E) does not imply (O).]

Nevertheless, this aside, it is clear that, in utilizing function-argument analysis and inventing the quantifier, Frege succeeded in developing the first system of predicate logic.

In Part II of the *Begriffsschrift*, Frege shows how more complex judgements can be represented and derived in his system. All we need note here is the 'kernel' of nine propositions that Frege takes as the axioms of his system (numbered (1), (2), (8), (28), (31), (41), (52), (54) and (58), respectively, in his own account):



As Lukasiewicz (1934) was later to prove, the first six axioms form a complete set of axioms for propositional logic, though the third axiom can in fact be derived, using just the rules of *modus ponens* and substitution, from the first two (cf. Kneale, 1962: pp. 490-1). The other five are independent of one another (cf. n. 9 of ch. 2 above). The axioms can be formulated in modern notation as follows:

(1*) $P \rightarrow (Q \rightarrow P)$.
(2*) $[R \rightarrow (Q \rightarrow P)] \rightarrow [(R \rightarrow Q) \rightarrow (R \rightarrow P)]$.
(3*) $[S \rightarrow (Q \rightarrow P)] \rightarrow [Q \rightarrow (S \rightarrow P)]$.
(4*) $(Q \rightarrow P) \rightarrow (\neg P \rightarrow \neg Q)$.
(5*) $\neg\neg P \rightarrow P$.
(6*) $P \rightarrow \neg\neg P$.

The seventh and eighth formulae involve Frege's symbol for identity of content, and might be re-expressed thus:

(7*) $(a = b) \rightarrow (Fa \rightarrow Fb)$.
(8*) $a = a$.

(7*) is a version of the Principle of the Indiscernibility of Identicals (cf. §4.2 above; making explicit the implicit quantification over properties, and replacing the second conditional by the biconditional, gives us '$a = b \rightarrow (\forall F)(Fa \leftrightarrow Fb)$'); and (8*), of course, states that everything is identical with itself. The ninth axiom, reflecting what, in a natural deduction system, is the rule of universal elimination, can be formulated as follows:

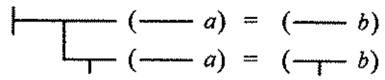
(9*) $(\forall x)\, Fx \rightarrow Fa$.

Frege also recognizes (*BS*, §11) what we now know as the rule of universal introduction, legitimizing the transition from a proposition involving italic letters to a universally quantified proposition (i.e. 'From *Fa* infer $(\forall x)\, Fx$' – on the understanding that '*a*' is an arbitrary name and *Fa* does not rest on any assumption in which '*a*' occurs). Here too Frege succeeded in specifying a complete set of axioms and rules for first-order predicate logic (cf. Kneale, p. 489).

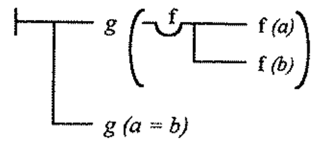
What revisions to his logical system were made in Frege's *Grundgesetze*? As noted in §4.2 above, and discussed in the chapters that follow, the two fundamental developments in Frege's philosophical views between the *Begriffsschrift* and the *Grundgesetze* concerned, firstly, the bifurcation of his early notion of 'content' into 'Sinn' and 'Bedeutung', and secondly, connected with this, the clarification of his ontology, admitting, in particular, both truth-values and extensions of concepts as objects, and hence as legitimate arguments of functions; and these developments did indeed necessitate certain changes in his logical theory. The essential system of 'strokes' remained the same, though Frege talks simply of the 'horizontal' rather than the 'content' stroke (cf. *GG*, I, §5). Taking truth-values as the *Bedeutungen* of sentences involved treating sentences as names, so that what is then seen as following the horizontal stroke is a name of a truth-value (cf. *GG*, I, §§2, 5). The main change – or addition – to the notation itself concerns the introduction of a symbol, '$\grave{\varepsilon}\Phi(\varepsilon)$', with a smooth breathing over the first occurrence of 'ε', for the *value-range* of the function $\Phi(\xi)$; where the function is one that maps objects onto one of the two truth-values, i.e. is a concept, then what we have here is a symbol for the extension of the concept Φ. (Cf. *GG*, I, §9.) As explained in chapter 4 above, Frege defines the natural numbers by identifying them with extensions of concepts that are logically definable; and by the time of the *Grundgesetze*, he had convinced himself that the appeal to extensions was legitimate (cf. §5.5 above).

As far as the axioms and rules of his logical system were concerned, there was a certain amount of reorganization. Axiom (1) of the *Be-*

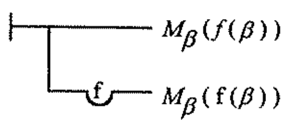
griffsschrift survives unchanged as Axiom I of the *Grundgesetze*, but Axioms (2), (3) and (4) disappear as a result of the specification of a greater number of rules. [18 rules are formulated altogether – see *GG*, I, §48 – but the last six simply concern the use of brackets, Rule 1 is a formation rule for the horizontal stroke, and Rules 9 to 12 are rules of substitution. Rules 2 to 8 are the rules of inference, in propositional and predicate logic.] Axiom (2) becomes provable by means of the rule of *amalgamation of identical subcomponents* (Rule 4; allowing e.g. the transition from 'if *P*, then if *P*, then *Q*' to the simpler 'if *P*, then *Q*'); and Axiom (3) becomes provable by means of the rule of *interchange of subcomponents* (Rule 2; allowing e.g. the transition from 'if *P*, then if *Q*, then *R*' to 'if *Q*, then if *P*, then *R*'). Axiom (4) is made redundant with the introduction of the rule of *contraposition* (Rule 3). Axioms (5) and (6), which, as Frege noted in the 'Preface' to the *Begriffsschrift*, he later realized could be combined using the symbol for identity of content (just as (5*) and (6*) above can be combined as '$\neg\neg P \leftrightarrow P$'), become derivable from Axiom IV of the *Grundgesetze*:



What this says is that either $a = b$ or $a \neq b$. Axioms (7) and (8) of the *Begriffsschrift* are replaced by Axiom III of the *Grundgesetze*:

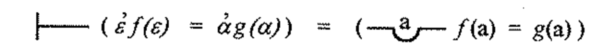


Given that both the expressions contained within the brackets (as governed by '*g*') are truth-values, what Axiom III says is that the truth-value named by the expression in the upper brackets falls under every concept under which the truth-value named by '$a = b$' falls (cf. *GG*, I, §20). From this it follows that if $a = b$, then anything that holds of *a* holds of *b*, which is Axiom (7) of the *Begriffsschrift*, and that everything is identical to itself, which is Axiom (8). Axiom (9) of the *Begriffsschrift* is retained unchanged as Axiom IIa of the *Grundgesetze*, but a corresponding version, Axiom IIb, is introduced for second-level functions:



What this says is that whatever holds of all first-level functions of one argument holds of any (cf. *GG*, I, §25). The nine axioms of the *Begriffsschrift* are thus condensed into the first four axioms of the *Grundgesetze*, with some additional rules formulated and with second-order quantification made official.

The most significant development, however, involved the introduction of Axiom V of the *Grundgesetze*, legitimizing value-ranges of functions (extensions of concepts). As the present work has been concerned to show, Axiom V played a central role in the evolution of Frege's philosophy: its origins lie in the central argument of the *Grundlagen* (see §5.3 above), the problem of its status motivated the distinction between *Sinn* and *Bedeutung* (see esp. §§5.4 – 5.5 above), and it was eventually held responsible by Frege for the contradiction that Russell discovered in his system (see §7.2 above). Frege formulated it as follows:



What this asserts is the equivalence between (Vb) and (Va), in effect contextually defining (Vb) by means of (Va):

(Va) The function *F* has the same value for each argument as the function *G*.

(Vb) The value-range of the function *F* is identical with the value-range of the function *G*.

In modern notation, adapting Frege's symbol for the value-range of a function, the equivalence can be expressed as follows:

(V*) $\grave{x} Fx = \grave{x} Gx \leftrightarrow (\forall x)(Fx \leftrightarrow Gx)$.

Frege's final axiom, Axiom VI, introduces a further new symbol, '\ξ', representing the function that Frege defines to replace the definite article of ordinary language. In ordinary language, definite descriptions, which Frege classifies as proper names, are formed by prefixing the definite article 'the' to a concept expression ('the *F*'). But such descriptions can readily be formed that lack a referent, or that fail to uniquely determine a single referent. Ordinary language is deficient in this respect, according to Frege, whereas in a logical language a referent must be determined for every legitimately constructed proper name. How might this be done? If we assume that concepts are sharply defined, and that extensions of concepts are objects, then any definite description of the form 'the extension of the concept *F*' is guaranteed a unique referent. So the obvious solution is to admit only definite descriptions of this form into the logical language. But how then do we refer to 'ordinary' objects? Here the strategy is to note that for any given object *Δ*, there is a concept *is identical with Δ*, which clearly meets the condition of being sharply defined. So the suggestion is to *identify* the object Δ with the extension of this concept (which is itself an object). However, as discussed in §6.4 above (and see esp. n. 29 of ch. 6, which states Frege's own argument in *GG*, I, §10), there are objections to identifying objects with the corresponding extensions (i.e. with their unit classes). So what Frege does instead is introduce a function that serves the same purpose, *mapping* extensions of concepts under which only one object falls onto that object itself. More precisely, the function $\backslash\xi$ is defined in the following way:

(i) If the argument is the extension of the concept *is identical with* Δ, i.e. $\grave{\varepsilon}(\Delta = \varepsilon)$, for any given object Δ, then the value of the function is the object Δ itself.

(ii) If the argument is not an extension as specified in (i), i.e. is not an extension of a concept under which only one object falls, then the value of the function is the argument itself (i.e. the function maps all other objects onto themselves). [Cf. *GG*, I, §11.]

(ii) ensures that the function is defined for all objects; and (i) gives rise to Axiom VI (cf. *GG*, I, §18):

$$\vdash a = \backslash\grave{\varepsilon}(a = \varepsilon)$$

Frege's strategy might be illustrated by considering the example that Russell used to motivate his theory of descriptions:

(K) The present King of France is bald.

If there is no King of France (i.e. in Fregean terminology, the definite description lacks a *Bedeutung*), then the proposition as a whole would seem to lack a truth-value (i.e. a *Bedeutung*). Yet (K) remains meaningful, and hence, according to Russell, *must* have a truth-value. To solve the problem, Russell suggests that (K) should be analysed as the following:

(KR) There is one and only one King of France, and whatever is King of France is bald.

The proposition now comes out as *false* because the first conjunct is false (there is no King of France). According to Frege, however, (K) should be rewritten in his ideal logical language thus:

(KF) \ (extension of the concept *present King of France*) is bald.

If there *is* one and only one present King of France, then the value of '\ (extension of the concept *present King of France*)' is that person; if there is not, then the value is the extension itself. If there is no present King of France at all, in other words, then the value is the null set, and it is false that the null set is bald. (If there were somehow more than one King of France, then the value would be the set of such people, and it is equally

false that this set is bald; although strictly speaking, of course, the vagueness of *ξ is bald* means that there is no genuine (i.e. logically correct) concept involved here, according to Frege; see §7.3 above.) So Frege's treatment has the same result as Russell's: propositions involving definite descriptions that fail to refer to a unique entity come out as *false*. The difference is that whilst Russell 'analysed away' the troublesome denoting phrase (cf. n. 36 of ch. 6 above), Frege ingeniously provided a replacement for it and introduced a technical device to ensure it had a referent (cf. the last two pages of §8.2 above). If, to use the term introduced in §5.3 above, Frege's central argument in the *Grundlagen*, in seeking to define the natural numbers in a logically impeccable way, relied on *conceptual ascent*, then his description function $\backslash\xi$ was required for *objectual descent* – to make the return trip from the pure logical realm of extensions of concepts to the everyday world of 'ordinary' objects.