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Proper Exclusion Right, Priority and Allocation of Positions*

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Abstract: Multiple positions will be allocated to a group of individuals without side payments. Every individual has preferences over the positions, can have at most one position and may behave strategically. The right of using each position relies on individuals' given priorities. We propose a new solution called the proper exclusion right core which always guarantees to have precisely one solution. The solution is efficient, weakly and properly fair, can be supported by competitive prices and easily found by a procedure in a strategy-proof way. It is built on a novel exclusion right system that respects priorities and maximizes self-consistent exclusion rights.

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1 Introduction

This paper aims at developing a new solution to the allocation problem of multiple positions to many individuals. These positions are often not private properties but community, public, or organization properties. Individuals may be entitled or obliged to take a position. Such problems arise in a variety of environments. For instance, positions of a public school or college must be allocated to students, offices/dormitories/committees must be assigned to faculty members/college students/legislators, and subsidized houses must be distributed to certain residents. Unlike the allocation problem of private commodities to which competitive prices are commonly accepted as an effective solution, there is no such a widely-accepted tool available for the allocation problem of non-private resources. Here both positions and individuals are indivisible and cannot be matched in fraction. No side payments will be involved.

Individuals each have their personal preferences over the positions and may not respond truthfully according to their preferences if it is not in their best interest. These positions will be assigned and no position can be used by two or more individuals, but the rights and preferences of individuals are often competing and conflict with each other. Usually, the right of using these positions by individuals is exogenously given by a priority structure. These priorities are typically determined by certain ad hoc, legal, or social rules or conventions and may reflect relative importance of individuals or their perceived need or entitlement (Hylland and Zeckhauser, 1979; Ergin, 2002; Abdulkadiroğlu and Sönmez, 2003; Kesten, 2006, 2010; Balbuzanov and Kotowski, 2019; Rong et al., 2020; Reny, 2022). Under such an environment, we want to address the following basic question: Is it possible to allocate these positions to the individuals competitively, efficiently, fairly and at the same time induce individuals to behave honestly? Ideally, we would love to achieve all these objectives. Unfortunately, we find that (Pareto) efficiency and fairness are not compatible with each other. In order to attain all other objectives, we replace fairness by weak and proper fairness. In this sense, we can give a positive and complete answer to the raised question.

The first key step in answering the question is to identify a proper range of exclusion rights of individuals to positions. An individual's exclusion right on a position is the right of the individual to exclude other individuals from using the position; see e.g., Hardin (1960); Ostrom (1990); Penner (1997); Merrill (1998); Smith (2012); Penner and Otsuka (2018). The right to exclude others is a basic principle of property. For instance, Penner (1996) advocates the right to exclude others as the right to use and the right to trade. Smith (2012) defines an exclusion strategy not just for private property but also for public property. The impact of the range of exclusion rights has been well documented in the tragedy of the commons and the tragedy of the anticommons.[1] The first tragedy is caused by overuse or over-exploitation of a commons basically due to lack of exclusion rights, whereas the second tragedy refers to underutilization of a commons because of too many exclusion rights. To overcome these difficulties, we have to find a proper range of exclusion rights by proposing a novel exclusion right system that respects the given priorities and maximizes self-consistent exclusion rights. We call this system the proper exclusion right system. It will be shown that this system always exists and is unique and defines a proper range of exclusion rights.

Our major contribution is the introduction of a new solution for the problem, called the proper exclusion right core. It is shown that this core always exists and surprisingly contains precisely one outcome, which is efficient, weakly and properly fair. This feature makes the new core radically different from many existing cores, which often provide multiple solutions or can be empty. The new core is built upon the proper exclusion right system and has strong explanatory and predictive power. For instance, we can easily show that on the one hand, when there are fewer exclusion rights than the proper exclusion right system has, the traditional core can contain many solutions and some of these solutions can be undesirable such as unfair, offering new insights into the first tragedy described previously, on the other hand, when there are more exclusion rights than the proper exclusion right system has, the traditional core can be empty, casting fresh light on the second tragedy. In contrast, our proper exclusion right core always guarantees to exist, provides a unique solution and eliminates undesirable allocations.

The concept of core has been extensively studied as a fundamental solution to various exchange and resource allocation problems with and without side payments.[2] As

[1] see e.g., Hardin (1968); Ostrom (1990); Heller (1998); Burger and Gochfeld (1998); Heller (2017); Buchanan and Yoon (2000); Frischmann et al. (2019); Meisinger (2022).

[2] See e.g., Gillies (1953); Debreu and Scarf (1963); Scarf (1967); Arrow and Hahn (1970); Shapley and Scarf (1974); Quinzii (1984); Demange and Gale (1985); Hildenbrand and Kirman (1988); Hildenbrand and

a prime notion of strategic equilibrium it prescribes a set of stable allocations that are immune to the threat of deviation by any coalition of individuals. We will also show that the unique exclusion right core allocation can be supported by competitive prices and easily found by the celebrated top trading cycle (TTC) mechanism in Shapley and Scarf (1974) with some modification. The competitive prices reveal relative strength of each individual in the economy and can be seen as a good measure of individuals' competitiveness, as competitive prices are widely accepted as a good measure of the value of private goods. We further demonstrate that when facing this TTC mechanism, it is in the best interest of every individual and every coalition of individuals to act honestly. We show that a mechanism is properly fair, Pareto efficient, and strategy-proof if and only if it is the TTC mechanism that finds the unique exclusion right core allocation.

We first build our theory on a one-to-one (i.e. unit-demand) model and then extend the analysis to a many-to-one model. For the extension, we take a general school choice problem as a prime example. In this problem, because every student attends at most one school, every school has multiple positions, and priorities of students are placed on every school not on positions of the school, inconsistency and ambiguities can easily arise in the exclusion right system and need to be overcome. We will show that a coherent approach can be found to resolve the issue.

We conclude this section by briefly reviewing related studies. Our work is closely related to two recent striking analyses of Balbuzanov and Kotowski (2019) and Reny (2022). Balbuzanov and Kotowski (2019) introduce an innovative concept of core, i.e., the exclusion core, based on exclusion rights for the exchange and allocation problems of indivisible goods with unit-demand agents. They reinterpret endowments of goods as a distribution of exclusion rights and establish several existence results. Compared with our unique proper core solution and unique proper exclusion right system, their exclusion cores may contain multiple solutions some of which are undesirable, their strong exclusion core can be empty, and their exclusion right system is not unique. Reny (2022) proposes an elegant solution of priority-efficiency for a priority-based school choice problem and establishes its existence. Like ours his solution is also unique but is based on two rights which are different from ours. It will be shown that his solution cannot

Sonnenschein (1991); Ma (1994); Abdulkadiroğlu and Sönmez (1998); Predtetchinski and Herings (2004).

achieve incentive compatibility nor can ensure weak fairness. In Section 5 we compare our solution with theirs in detail. Early important papers on the assignment of indivisible objects with unit-demand agents include Koopmans and Beckmann (1957); Shapley and Shubik (1971); Shapley and Scarf (1974); Hylland and Zeckhauser (1979); Crawford and Knoer (1981); Ma (1994); Sönmez (1999); Pápai (2000); Ergin (2002).

Dur and Morrill (2018) study competitive equilibrium in a school assignment problem and show that every competitive equilibrium with weakly decreasing prices induces a unique allocation that can be produced by the TTC mechanism. Sun et al. (2020) examine markets with co-ownership and indivisibility and propose an effective core to address the inadequacies of the conventional core. Zhang (2020) considers discrete exchange economics with possibly redundant and joint ownership. He proposes an induction core by identifying self-enforcing coalitions to overcome the shortcomings of the conventional core. Rong et al. (2020) introduce two concepts of core for a priority-based school choice problem and study their properties. Balbuzanov and Kotowski (2021) generalize their exclusion core to a production economy and find sufficient conditions for the existence of ex ante and ex post exclusion cores. Sun and Yang (2021) study stable and core allocations in senior job matching markets with commitment.

The rest of this paper is organized as follows. Section 2 introduces the basic model and the proper exclusion right core and discusses its properties. Section 3 presents the mechanisms for finding the proper exclusion right system, the proper exclusion right core and examines their properties. Section 4 discusses an extended many-to-one model. Section 5 makes a comparison. Section 6 concludes.

2 Model and Solution Concepts

2.1 Model

There are two finite and disjoint sets I of agents and S of indivisible objects with $I = \{i_1, \ldots, i_{|I|}\}$ and $S = \{s_1, \ldots, s_{|S|}\}$. Objects can be positions or houses. Let $s_0 \notin S$ be a dummy item, i.e., the outside option of the agents and let i_0 be the virtual agent. Each agent demands at most one position and each position may take in one agent (i.e., unit-demand or one-to-one). Each agent i has a strict, complete, and transitive preference relation $\succ_i$ on objects in $S \cup \{s_0\}$. We write $s \succ_i s'$ if agent i strictly prefers s to s', and $s \succeq_i s'$ if $s \succ_i s'$ or $s = s'$. Let $\mathcal{P}^i$ denote the set of the agent's all preference relations. Each object $s \in S$ has strict, complete, and transitive priorities over agents in $I \cup \{i_0\}$. We write $i \rhd_s j$ if agent i has a higher priority on object s than agent j, and $i \unrhd_s j$ if $i \rhd_s j$ or $i = j$. It is reasonable to assume that for any object $s \in S$, real agents always have higher priorities than the virtual agent, i.e., $i \rhd_s i_0$ for all $i \in I$. Let $\succ = (\succ_i)_{i \in I}$ be the preference profile of all agents, $\mathcal{P}^I = \prod_{i \in I} \mathcal{P}^i$ the set of all preference profiles, and $\rhd = (\rhd_s)_{s \in S}$ the priority structure. A relational economy is a tuple $\langle I, S, \succ, \rhd \rangle$ and a relational environment is a triple $\langle I, S, \rhd \rangle$ without the preferences of agents.

An allocation is a function $\mu : I \to S \cup \{s_0\}$ such that $|\mu^{-1}(s)| \leq 1$ for each $s \in S$. For any $s \in S$, if there is no agent $i \in I$ such that $\mu(i) = s$, we will write $\mu^{-1}(s) = i_0$. If $\mu(i) = s_0$, we say that agent i is unassigned. For simplicity, we use $\mu(C)$ to denote $\bigcup_{i \in C} \mu(i)$. Let $\mathcal{A}$ be the set of all allocations.

2.2 Exclusion right and core

With respect to any given allocation $\mu \in \mathcal{A}$, we introduce a binary relation $\blacktriangleright_\mu$ on the set of agents. We say that agent i **has a (direct) right** to exclude agent j from her object $\mu(j)$ if $i \blacktriangleright_\mu j$. In this case, agent i is the right holder, agent j is the occupant, and μ is the executive condition. An agent i **has an (indirect) right** to exclude agent j from her object $\mu(j)$ if there is a nonempty sequence of agents $\{i_1, \ldots, i_L\}$ such that $i \blacktriangleright_\mu i_1 \blacktriangleright_\mu \cdots \blacktriangleright_\mu i_L \blacktriangleright_\mu j$. We write $i \blacktriangleright\!\!\blacktriangleright_\mu j$ to denote that agent i has a right to directly or indirectly exclude agent j. Note that an unassigned agent cannot be excluded by anyone. It is allowed that i has a right to exclude herself, i.e., $i \blacktriangleright\!\!\blacktriangleright_\mu i$. However, a rational agent will not exercise her exclusion right to herself.

A **direct (exclusion right) scheme** $\blacktriangleright = (\blacktriangleright_\mu)_{\mu \in \mathcal{A}}$ prescribes who has a right to directly exclude whom at each possible allocation $\mu \in \mathcal{A}$. Let $\blacktriangleright\!\!\blacktriangleright = (\blacktriangleright\!\!\blacktriangleright_\mu)_{\mu \in \mathcal{A}}$ be the **derived (exclusion right) scheme** from the direct scheme $\blacktriangleright$. Sometimes, for ease of notation in the derived scheme, we use $i \blacktriangleright_\mu j$ instead of $i \blacktriangleright\!\!\blacktriangleright_\mu j$ when i has a direct right to exclude j at μ. An **exclusion right system** $(\blacktriangleright, \blacktriangleright\!\!\blacktriangleright)$ consists of a direct scheme $\blacktriangleright$ and a derived scheme $\blacktriangleright\!\!\blacktriangleright$. We will use superscripts to distinguish different systems. For example, the scheme $\blacktriangleright\!\!\blacktriangleright$ without a superscript is derived from $\blacktriangleright$, and the scheme $\blacktriangleright\!\!\blacktriangleright^p$ with a superscript p is derived from $\blacktriangleright^p$. We use an example to explain the above concepts.

Example 1 *Let* $I = \{i_1, i_2, i_3\}$ *and* $S = \{s_1, s_2\}$. *The priorities of the objects and the preferences of agents are given by*

$$\rhd_{s_1}: i_3, i_1, i_2 \quad \rhd_{s_2}: i_1, i_2, i_3 \qquad \succ_{i_1}: s_1, s_2, s_0 \quad \succ_{i_2}: s_1, s_2, s_0 \quad \succ_{i_3}: s_2, s_1, s_0$$

Table 1: The exclusion right schemes

μ	i_1 i_2 i_3	$\blacktriangleright$	$\blacktriangleright\!\!\blacktriangleright$
μ_1	s_1 s_2 s_0	$i_3 \blacktriangleright_{\mu_1} i_1, i_1 \blacktriangleright_{\mu_1} i_1$ $i_1 \blacktriangleright_{\mu_1} i_2, i_2 \blacktriangleright_{\mu_1} i_2$	$i_3 \blacktriangleright_{\mu_1} i_1, i_1 \blacktriangleright_{\mu_1} i_1$ $i_1 \blacktriangleright_{\mu_1} i_2, i_2 \blacktriangleright_{\mu_1} i_2, i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_1} i_2$
μ_2	s_1 s_0 s_2	$i_3 \blacktriangleright_{\mu_2} i_1, i_1 \blacktriangleright_{\mu_2} i_1$ $i_1 \blacktriangleright_{\mu_2} i_3, i_2 \blacktriangleright_{\mu_2} i_3, i_3 \blacktriangleright_{\mu_2} i_3$	$i_3 \blacktriangleright_{\mu_2} i_1, i_1 \blacktriangleright_{\mu_2} i_1, i_2 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_1$ $i_1 \blacktriangleright_{\mu_2} i_3, i_2 \blacktriangleright_{\mu_2} i_3, i_3 \blacktriangleright_{\mu_2} i_3$
μ_3	s_2 s_1 s_0	$i_1 \blacktriangleright_{\mu_3} i_1, i_3 \blacktriangleright_{\mu_3} i_2, i_1 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$	$i_1 \blacktriangleright_{\mu_3} i_1, i_3 \blacktriangleright_{\mu_3} i_2, i_1 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$
μ_4	s_2 s_0 s_1	$i_1 \blacktriangleright_{\mu_4} i_1, i_3 \blacktriangleright_{\mu_4} i_3$	$i_1 \blacktriangleright_{\mu_4} i_1, i_3 \blacktriangleright_{\mu_4} i_3$
μ_5	s_0 s_1 s_2	$i_3 \blacktriangleright_{\mu_5} i_2, i_1 \blacktriangleright_{\mu_5} i_2, i_2 \blacktriangleright_{\mu_5} i_2$ $i_1 \blacktriangleright_{\mu_5} i_3, i_2 \blacktriangleright_{\mu_5} i_3, i_3 \blacktriangleright_{\mu_5} i_3$	$i_3 \blacktriangleright_{\mu_5} i_2, i_1 \blacktriangleright_{\mu_5} i_2, i_2 \blacktriangleright_{\mu_5} i_2$ $i_1 \blacktriangleright_{\mu_5} i_3, i_2 \blacktriangleright_{\mu_5} i_3, i_3 \blacktriangleright_{\mu_5} i_3$
μ_6	s_0 s_2 s_1	$i_1 \blacktriangleright_{\mu_6} i_2, i_2 \blacktriangleright_{\mu_6} i_2, i_3 \blacktriangleright_{\mu_6} i_3$	$i_1 \blacktriangleright_{\mu_6} i_2, i_2 \blacktriangleright_{\mu_6} i_2, i_3 \blacktriangleright_{\mu_6} i_3$
μ_7	s_1 s_0 s_0	$i_3 \blacktriangleright_{\mu_7} i_1, i_1 \blacktriangleright_{\mu_7} i_1$	$i_3 \blacktriangleright_{\mu_7} i_1, i_1 \blacktriangleright_{\mu_7} i_1$
μ_8	s_2 s_0 s_0	$i_1 \blacktriangleright_{\mu_8} i_1$	$i_1 \blacktriangleright_{\mu_8} i_1$
μ_9	s_0 s_1 s_0	$i_3 \blacktriangleright_{\mu_9} i_2, i_1 \blacktriangleright_{\mu_9} i_2, i_2 \blacktriangleright_{\mu_9} i_2$	$i_3 \blacktriangleright_{\mu_9} i_2, i_1 \blacktriangleright_{\mu_9} i_2, i_2 \blacktriangleright_{\mu_9} i_2$
μ_{10}	s_0 s_2 s_0	$i_1 \blacktriangleright_{\mu_{10}} i_2, i_2 \blacktriangleright_{\mu_{10}} i_2$	$i_1 \blacktriangleright_{\mu_{10}} i_2, i_2 \blacktriangleright_{\mu_{10}} i_2$
μ_{11}	s_0 s_0 s_1	$i_3 \blacktriangleright_{\mu_{11}} i_3$	$i_3 \blacktriangleright_{\mu_{11}} i_3$
μ_{12}	s_0 s_0 s_2	$i_1 \blacktriangleright_{\mu_{12}} i_3, i_2 \blacktriangleright_{\mu_{12}} i_3, i_3 \blacktriangleright_{\mu_{12}} i_3$	$i_1 \blacktriangleright_{\mu_{12}} i_3, i_2 \blacktriangleright_{\mu_{12}} i_3, i_3 \blacktriangleright_{\mu_{12}} i_3$
μ_{13}	s_0 s_0 s_0	$\emptyset$	$\emptyset$

Priorities among various things such as ownership, endowments, urgency, and degree of need or importance can be used to define exclusion rights. For a given priority structure, we can grant a direct exclusion right on an object to an agent who has a weakly higher priority than the occupant of the object.[3] Table 1 shows the direct exclusion right scheme defined in this way. For example, s_1's occupant at μ_1 is i_1 so any agent who has a weakly higher priority on s_1 than i_1, including i_3 and i_1, has a direct exclusion right to i_1, i.e. $i_3 \blacktriangleright_{\mu_1} i_1$ and $i_1 \blacktriangleright_{\mu_1} i_1$. Let us see how indirect exclusion rights come. At μ_1, because i_3 has a direct exclusion right to i_1 who has a direct exclusion right to i_2, i_3 can ask i_1 to exclude i_2 by threatening i_1 to evict her from s_1. So we see that i_3 has an indirect right to exclude i_2 on s_2, i.e., $i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_1} i_2$.

[3]This is also the weak conditional endowment defined by Balbuzanov and Kotowski (2019).

We now adapt the classical concept of core to the environment of exclusion rights. A nonempty subset of the set I of agents is called a coalition.

Definition 1 *Given the derived exclusion right scheme* $\blacktriangleright\!\!\!\blacktriangleright$*, an allocation* $\mu \in \mathcal{A}$ *is blocked by a coalition* $C \subseteq I$ *if there exists another allocation* $\nu \in \mathcal{A}$ *such that* $\nu(i) \succ_i \mu(i)$ *for all* $i \in C$ *and* $\mu(j) \succ_j \nu(j)$ *implies that there is an agent* $i \in C$ *such that* $i \blacktriangleright\!\!\!\blacktriangleright_\mu j$*. The* ***(exclusion right) core*** *is the set of allocations that cannot be blocked by any coalition.*

An allocation μ can be blocked by a coalition C if there exists a different allocation ν such that every coalition member in C gets better off and anyone who gets worse off is directly or indirectly excluded from their objects by a member of the coalition C. The core just defined is very similar to the traditional core. We will first show how the exclusion right scheme can influence the core outcomes.

The following simple example shows that if a relational economy has no or few exclusion rights, the core can contain too many solutions some of which are undesirable, while if a relational economy has too many exclusion rights, the core can be empty. The former case offers a fresh economic understanding of the famous tragedy of the commons by Hardin (1968) who argues that a commons or a publicly owned resource can be over-exploited if there are no or just few exclusion rights, which can easily result in no or little control of the use of the resources, whereas the latter case gives new economic insights into the well-known tragedy of the anticommons by Heller (1998) who shows that a commons can be severely underutilized if there are too many exclusion rights, which can easily lead to no agreement or solution at all.

Example 2 *There are two agents and one object with* $I = \{i, j\}$ *and* $S = \{s\}$*. We have priorities* $\rhd_s$: i, j*. Both agents prefer* s *to* s_0*.*

Table 2: Four exclusion right schemes

μ	i	j	$\blacktriangleright\!\!\!\blacktriangleright^1$	$\blacktriangleright\!\!\!\blacktriangleright^2$	$\blacktriangleright\!\!\!\blacktriangleright^3$	$\blacktriangleright\!\!\!\blacktriangleright^*$
μ_0	s_0	s_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
μ_1	s	s_0	$\emptyset$	$i \blacktriangleright_{\mu_1} i, j \blacktriangleright_{\mu_1} i$	$i \blacktriangleright_{\mu_1} i$	$i \blacktriangleright_{\mu_1} i$
μ_2	s_0	s	$\emptyset$	$i \blacktriangleright_{\mu_2} j, j \blacktriangleright_{\mu_2} j$	$j \blacktriangleright_{\mu_2} j$	$i \blacktriangleright_{\mu_2} j, j \blacktriangleright_{\mu_2} j$

This economy has three feasible allocations as shown in Table 2. We examine four different derived exclusion right schemes given in the table. Let us first look at the derived exclusion right scheme $\blacktriangleright^1$. This scheme contains no exclusion right, i.e., no agent has any exclusion right. Clearly, both μ_1 and μ_2 are in the core under $\blacktriangleright^1$. Unfortunately, μ_2 is unfair to agent i, because i prefers s to s_0 and also has a higher priority than agent j who is assigned s at μ_2.

Now we look at the derived exclusion right scheme $\blacktriangleright^2$. This scheme contains so many exclusion rights that the core fails to exist. More precisely, μ_0 can be blocked by either $\{i\}$ or $\{j\}$ because object s is unoccupied and no one is hurt by the blocking. Allocation μ_1 is blocked by coalition $\{j\}$ because $s \succ_j \mu_1(j) = s_0$ and $j \blacktriangleright_{\mu_1} i$. Allocation μ_2 is blocked by coalition $\{i\}$ because $s \succ_i \mu_2(i) = s_0$ and $i \blacktriangleright_{\mu_2} j$.

This example shows that the derived exclusion right scheme can have a huge impact on the core outcomes. In the next section, we discuss how to construct a proper exclusion right system for every given relational economy. We will see what solution can be offered to this example.

2.3 Proper exclusion right and proper core

Our ultimate goal is to allocate objects to agents in an efficient, fair, competitive, and incentive compatible way. To achieve this, we will introduce a proper exclusion right system and a proper exclusion right core for any given relational economy. We first discuss two intuitive and plausible properties for any given exclusion right system.

In a relational economy with a priority structure, a natural requirement of a direct exclusion right scheme is to reflect the given priorities. This principle has been explored by Balbuzanov and Kotowski (2019); Ergin (2002); Kesten (2006); Reny (2022) among others and widely used in practice in various forms. Given an allocation μ, the direct exclusion right relation $\blacktriangleright_\mu$ **respects the priority structure** $\triangleright$, if for any three agents $i, j, k \in I$ such that $\mu(i) \in S$, we have

(A1) $j \blacktriangleright_\mu i$ only if $j \trianglerighteq_{\mu(i)} i$, and

(A2) if $k \trianglerighteq_{\mu(i)} j \blacktriangleright_\mu i$, then $k \blacktriangleright_\mu i$.

Requirement (A1) states that exclusion rights are only granted to those with higher priorities. Requirement (A2) says that if at allocation μ, agent j has a direct exclusion right to agent i and another agent k has a higher priority than agent j on agent i's assignment $\mu(i)$, then agent k also has a direct exclusion right to agent i. The next one is the first key property for a direct exclusion right scheme.

Definition 2 *A direct exclusion right scheme $\blacktriangleright = (\blacktriangleright_\mu)_{\mu \in \mathcal{A}}$ **respects priorities** $\triangleright$ if the direct exclusion right relation $\blacktriangleright_\mu$ respects $\triangleright$ for every allocation $\mu \in \mathcal{A}$. We call such a scheme **the priority respecting direct exclusion right scheme**.*

The next property concerns how to distribute the exclusion rights in a consistent and coherent way.

Definition 3 *The derived exclusion right scheme $\blacktriangleright\!\!\blacktriangleright = (\blacktriangleright\!\!\blacktriangleright_\mu)_{\mu \in \mathcal{A}}$ has **contradictory rights** if there exist two different agents $i, j \in I$ and two different allocations $\mu, \nu \in \mathcal{A}$ such that $\mu(i) = \nu(j) = s \in S$, $\mu(k) = \nu(k)$ for every other agent $k \in I \setminus \{i, j\}$, and $j \blacktriangleright\!\!\blacktriangleright_\mu i \blacktriangleright\!\!\blacktriangleright_\nu j$. The scheme $\blacktriangleright\!\!\blacktriangleright$ is **self-consistent** if there are no contradictory rights.*

In other words, contradictory rights occur if there exist two agents i, j who can exclude each other from an object s without changing the assignments of other agents. As a result, there may not be a proper way to allocate objects like s when the derived exclusion right scheme has contradictory rights. We use Example 1 to illustrate this point. Consider the two allocations μ_1 and μ_2, where $\mu_1(i_2) = \mu_2(i_3) = s_2$ and $\mu_1(i_1) = \mu_2(i_1) = s_1$. Suppose that the derived exclusion right scheme at μ_1 and μ_2 are $\blacktriangleright\!\!\blacktriangleright_{\mu_1}$ and $\blacktriangleright\!\!\blacktriangleright_{\mu_2}$, respectively, as shown in Table 1. Clearly, the derived scheme has contradictory rights with $i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_1} i_2 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_3$. Let us see how these contradictory rights create a hurdle to a proper assignment of objects. Object s_2 cannot be properly allocated because if the object is assigned to agent i_1, agent i_2 will exclude agent i_1 from the object, and if the object is assigned to agent i_2, agent i_1 will exclude agent i_2 from the object too.

Proposition 1 *The core can be empty if the derived exclusion right scheme $\blacktriangleright\!\!\blacktriangleright$ is **not** self-consistent.*

Self-consistency is a necessary condition to ensure the existence of the core in the relational economy with a priority structure. However, not every self-consistent derived exclusion right scheme is proper. For instance, if a derived exclusion right scheme (like the scheme $\blacktriangleright\!\!\!>^1$ in Example 2) is empty, it is self-consistent. Under this scheme, priorities do not work at all because the exclusion rights distributed by $\blacktriangleright\!\!\!>$ do not depend on priorities. To avoid this situation, a natural requirement of a proper scheme is to distribute as many exclusion rights as possible, as long as the scheme is self-consistent. We say that a derived exclusion right scheme $\blacktriangleright\!\!\!>'$ is **larger** than $\blacktriangleright\!\!\!>$ if $i \blacktriangleright\!\!\!>_\mu j$ implies $i \blacktriangleright\!\!\!>'_\mu j$ for all $\mu \in \mathcal{A}$ and all $i, j \in I$ and we have $i \blacktriangleright\!\!\!>'_\nu j$ but not $i \blacktriangleright\!\!\!>_\nu j$ for at least one allocation $\nu \in \mathcal{A}$ and two agents $i, j \in I$.

Definition 4 *A derived exclusion right scheme $\blacktriangleright\!\!\!>$ has **maximal self-consistent (MAXISC) exclusion rights** if it is self-consistent but any larger derived exclusion right scheme $\blacktriangleright\!\!\!>'$ is not self-consistent.*

The self-consistent derived exclusion right scheme $\blacktriangleright\!\!\!>$ discussed before this definition is empty and clearly does not have MAXISC exclusion rights. A larger derived exclusion right scheme $\blacktriangleright\!\!\!>$ can be that for all $\mu \in \mathcal{A}$ and for all $i \in I$ such that $\mu(i) \in S$, $i \blacktriangleright\!\!\!>'_\mu i$. The larger derived exclusion right scheme $\blacktriangleright\!\!\!>'$ is self-consistent because no one will exercise exclusion right to herself and no contradictory right forms. The following lemma states that every assigned agent has a right to exclude herself under $\blacktriangleright\!\!\!>$ is a necessary condition for $\blacktriangleright\!\!\!>$ to have MAXISC exclusion rights.

Lemma 1 *A derived exclusion right scheme $\blacktriangleright\!\!\!>$ has MAXISC exclusion rights only if, for every $\mu \in \mathcal{A}$ and every $i \in I$, $\mu(i) \in S$ implies $i \blacktriangleright\!\!\!>_\mu i$.*

Let us revisit Example 2. Look at the scheme $\blacktriangleright\!\!\!>^*$ in Table 2 which has MAXISC exclusion rights. We will first show that adding one more exclusion right to the exclusion right scheme $\blacktriangleright\!\!\!>^*$ can create contradictory rights, which may cause the nonexistence of the core. The derived exclusion right scheme $\blacktriangleright\!\!\!>^2$ in the table has one more exclusion right $j \blacktriangleright_{\mu_1} i$ than $\blacktriangleright\!\!\!>^*$. As shown in Example 2, the core under $\blacktriangleright\!\!\!>^2$ is empty. Next, we will show that reducing one exclusion right $i \blacktriangleright_{\mu_2} j$ from the scheme $\blacktriangleright\!\!\!>^*$ will create undesirable outcomes in the core. The derived exclusion right scheme $\blacktriangleright\!\!\!>^3$ has one less exclusion right than $\blacktriangleright\!\!\!>^*$. In this case, allocations μ_1 and μ_2 are in the core under $\blacktriangleright\!\!\!>^3$ but

μ_2 is unfair to agent i who prefers s to s_0 and has a higher priority than agent j who is assigned s.

Having the above discussion, we can now introduce the concept of proper exclusion right system.

Definition 5 *An exclusion right system $(\blacktriangleright, \blacktriangleright\!\!\!\blacktriangleright)$ is* ***proper*** *if the derived exclusion right scheme $\blacktriangleright\!\!\!\blacktriangleright$ from the priority respecting direct exclusion right scheme $\blacktriangleright$ maximizes self-consistent exclusion rights. The scheme $\blacktriangleright\!\!\!\blacktriangleright$ is called* ***a proper derived exclusion right scheme*** *if the system $(\blacktriangleright, \blacktriangleright\!\!\!\blacktriangleright)$ is proper.*

The proper exclusion right core we propose here is defined on the proper system.

Definition 6 *An allocation is in the* ***proper (exclusion right) core*** *if it is not blocked by any coalition given the proper exclusion right system.*

The economy given in Example 2 has a unique proper exclusion right core allocation μ_1 and the unfair core allocation μ_2 is eliminated. In general, we have the following theorem on the existence of a proper derived exclusion right scheme, a proper exclusion right core allocation, and their uniqueness. In the remaining part of this paper, we use $\blacktriangleright\!\!\!\blacktriangleright^p$ to denote the unique proper derived exclusion right scheme and $\blacktriangleright^p$ the corresponding direct exclusion right scheme.

Theorem 1 *Every relational environment $\langle I, S, \rhd \rangle$ has a unique proper derived exclusion right scheme $\blacktriangleright\!\!\!\blacktriangleright^p$; every relational economy $\langle I, S, \succ, \rhd \rangle$ has a unique proper exclusion right core allocation.*

We now examine several important properties of the proper exclusion right core allocation.

An allocation $\mu \in \mathcal{A}$ is **Pareto dominated** by another allocation $\nu \in \mathcal{A}$ if $\nu(i) \succeq \mu(i)$ for all $i \in I$ and $\nu(i) \succ \mu(i)$ for some $i \in I$. An allocation $\mu \in \mathcal{A}$ is **(Pareto) efficient** if it is not Pareto dominated by any other allocation. An allocation $\mu \in \mathcal{A}$ is **individually rational** if $\mu(i) \succeq_i s_0$ for every $i \in I$.

Proposition 2 *For every relational economy $\langle I, S, \succ, \rhd \rangle$, the unique proper exclusion core allocation is efficient and also individually rational.*

Fairness is a fundamental criterion for the distribution of welfare and resources, especially important for public or community-owned resources.[4] We will adapt this concept to the current model. We say that agent i justly envies agent j at allocation μ if $\mu(j) \succ_i \mu(i)$ and $i \rhd_{\mu(j)} j$. That is, agent i prefers agent j's assignment to her own assignment and has a higher priority on agent j's assignment than agent j. An allocation is **fair** if no agent justly envies any other.

Unfortunately, the notion of fairness is too strong to be compatible with efficiency. Let us revisit Example 1. There are 13 feasible allocations given in Table 1. In fact, μ_4 is the unique fair allocation but not efficient, because it is Pareto dominated by allocation μ_2. All of μ_1, μ_2, μ_3, and μ_5 are efficient. This shows that fairness and efficiency are incompatible.

We introduce two weaker and more plausible notions of fairness. Given the proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\!\blacktriangleright^p)$, we say that agent i **properly envies** agent j at μ if $\mu(j) \succ_i \mu(i)$ and $i \blacktriangleright^p_\mu j$. That is, agent i prefers agent j's assignment to her own assignment and has a right to exclude j. We say that agent i **strongly envies** agent j at μ if $\mu(j) \succ_i \mu(i)$ and $i \rhd_s j$ for all $s \in S$. That is, agent i prefers agent j's assignment to her own assignment and has a higher priority right than agent j has on every object.

Definition 7 *An allocation is* ***properly fair*** *if no agent properly envies any other agent. An allocation is* ***weakly fair*** *if no agent strongly envies any other agent.*

In general, we have the following result.

Proposition 3 *For every relational economy $\langle I, S, \succ, \rhd \rangle$, the unique proper exclusion right core allocation is both properly and weakly fair.*

Next, we adapt the fundamental concept of competitive equilibrium to the current model. See Dur and Morrill (2018) on the application of this concept to a school assignment problem. Let $p \in \mathbb{R}^{S \cup \{s_0\}}$ be a price vector indicating a price $p(s)$ for every object $s \in S \cup \{s_0\}$ with $p(s_0) = 0$ for the dummy object. Let $y \in \mathbb{R}^I$ be an income vector that indicates income $y(i)$ for every agent $i \in I$. Given an allocation μ and a price

[4] See e.g., Foley (1967); Rawls (1971); Varian (1974); Abdulkadiroğlu and Sönmez (2003); Sun and Yang (2003); Kesten and Yazici (2012).

vector p, we say that the income vector y is consistent with the direct exclusion right relation $\blacktriangleright_\mu$ if $y(i) = \max\{p(s) \mid s = \mu(j) \text{ for some } j \text{ such that } i \blacktriangleright_\mu j\}$. Given a price vector p and a consistent income vector y, we define the budget set of agent $i \in I$ as $B^i(p,y) = \{s \in S \cup \{s_0\} \mid p(s) \leq y(i)\}$ and the demand set of the agent as

$$D^i(p,y) = \left\{s \in B^i(p,y) \mid s \succeq_i s' \text{ for all } s' \in B^i(p,y)\right\}.$$

A competitive equilibrium (p, y, μ) consists of (1) a price vector p at which $p(s) = 0$ for every unassigned object $s \in S \setminus \mu(I)$, (2) an income vector y being consistent with $\blacktriangleright_\mu$, and (3) an allocation μ at which $\mu(i) \in D^i(p,y)$ for every agent $i \in I$. We call the allocation μ a competitive allocation and the vector p competitive equilibrium prices. We also say that μ is supported by competitive prices.

Proposition 4 *The proper exclusion right core allocation of every relational economy $\langle I, S, \succ, \rhd \rangle$ is also a competitive equilibrium allocation under the proper exclusion right system.*

3 Mechanisms

In this section, we introduce two mechanisms and examine their properties. The first mechanism is designed to find a proper exclusion right system and the second one is proposed to find a proper exclusion right core allocation.

3.1 A mechanism for a proper exclusion right system

We first construct a proper exclusion right system from any given priority structure. To achieve this goal, we introduce a threshold function $\theta_\mu : I \to I \cup \{\emptyset\}$ to represent the direct exclusion right scheme $\blacktriangleright_\mu$ such that for every $i \in I$, (1) $\theta_\mu(i) \trianglerighteq_{\mu(i)} i$, and (2) $j \blacktriangleright_\mu i$ if and only if $j \trianglerighteq_{\mu(i)} \theta_\mu(i)$. That's to say, the threshold of agent i has a weakly higher priority of object $\mu(i)$ than i. Any agent who has a weakly higher priority than the threshold has a direct exclusion right to i and any agent who has a lower priority than the threshold does not have a direct exclusion right to i. The case of $\theta_\mu(i) = \emptyset$ means that no agent has an exclusion right to i. Let $\theta = (\theta_\mu)_{\mu \in \mathcal{A}}$ be a threshold scheme. We have the following result.

Proposition 5 *A direct exclusion right scheme $\blacktriangleright = (\blacktriangleright_\mu)_{\mu \in \mathcal{A}}$ respects priorities $\rhd$ if and only if there exists a threshold scheme $\theta = (\theta_\mu)_{\mu \in \mathcal{A}}$ that represents the scheme $\blacktriangleright$.*

The proper exclusion right system respects the priority structure $\rhd$. Therefore, the system can be characterized by a threshold scheme. Now we propose a method called the Top Priority Cycle (TPC) algorithm to find a threshold function θ_μ^g for every allocation $\mu \in \mathcal{A}$ so obtain a threshold scheme $\theta^g = (\theta_\mu^g)_{\mu \in \mathcal{A}}$. We further obtain the derived exclusion right scheme $\mathrel{\blacktriangleright\!\!\!\blacktriangleright}^g$ of the threshold scheme θ^g. We call $\mathrel{\blacktriangleright\!\!\!\blacktriangleright}^g$ the TPC-derived exclusion right scheme.

Top Priority Cycle Algorithm

- For any given allocation μ, remove all agents in $I^0 = \{i \in I \mid \mu(i) = s_0\}$ and all objects in $S^0 = \{s \in S \mid \mu^{-1}(s) = i_0\}$. For every $i \in I^0$, set $\theta_\mu^g(i) = \emptyset$. Then set $t = 1$, $I^1 = I \setminus I^0$, and $S^1 = S \setminus S^0$.
- At each step $t \geq 1$, every remaining agent $i \in I^t$ points to $\mu(i)$. Every remaining object $s \in S^t$ points to the remaining agent who has the highest priority on s among agents in I^t. There exists at least one cycle. Let X^t be the set of agents and objects involved in cycles at this step. For every agent $i \in X^t$, set $\theta_\mu^g(i)$ to be the agent to which $\mu(i)$ points. Remove all cycles by setting $I^{t+1} = I^t \setminus X^t$ and $S^{t+1} = S^t \setminus X^t$. Set $t = t + 1$ and repeat the operation until all agents and objects are removed.

The following theorem shows that the TPC-derived exclusion right scheme $\mathrel{\blacktriangleright\!\!\!\blacktriangleright}^g$ is proper and unique.

Theorem 2 *An exclusion right system $(\blacktriangleright^p, \mathrel{\blacktriangleright\!\!\!\blacktriangleright}^p)$ is proper if and only if the direct exclusion right scheme $\blacktriangleright^p$ respects $\rhd$ and the derived exclusion right scheme $\mathrel{\blacktriangleright\!\!\!\blacktriangleright}^p$ equals the TPC-derived exclusion right scheme $\mathrel{\blacktriangleright\!\!\!\blacktriangleright}^g$.*

The next proposition states that if two allocations μ and ν differ in the assignment of two agents i and j with $\mu(i) = \nu(j) = s$, then only one of the two agents has an exclusion right to object s.

Proposition 6 *Under any given proper exclusion right system $(\blacktriangleright^p, \mathrel{\blacktriangleright\!\!\!\blacktriangleright}^p)$, for any two different allocations $\mu, \nu \in \mathcal{A}$ and any two different agents $i, j \in I$ satisfying $\mu(i) = \nu(j) = s \in S$ and $\mu(k) = \nu(k)$ for all other agents $k \in I \setminus \{i, j\}$, we have either $i \mathrel{\blacktriangleright\!\!\!\blacktriangleright}_\nu^p j$ or exclusively $j \mathrel{\blacktriangleright\!\!\!\blacktriangleright}_\mu^p i$.*

3.2 A mechanism for the proper exclusion right core

We will adapt the famous top trading cycle (TTC) algorithm of Shapley and Scarf (1974) to our current model to find a proper exclusion right core allocation, denoted by μ^*, for any given relational economy.

Top Trading Cycle Algorithm

- Let $I^1 = I$, $S^1 = S$, and $t = 1$.
- At each step $t \geq 1$, every remaining agent $i \in I^t$ points to the object most preferred by her among objects in $S^t \cup \{s_0\}$. Every remaining object points to the $\triangleright$-maximal agent in I^t.
  - If the set of agents who point to s_0 is not empty, let X^t be this set. Every agent $i \in X^t$ leaves with assignment $\mu^*(i) = s_0$. Then let $I^{t+1} = I^t \setminus X^t$. Set $t = t + 1$. Go to the next step.
  - Otherwise, there exists at least one cycle. Let X^t be the set of agents and objects in the cycles. Assign every agent $i \in X^t$ the object to which she points in a cycle and let $\mu^*(i) = s$. All agents and objects in X^t leave. Set $I^{t+1} = I^t \setminus X^t$ and $S^{t+1} = S^t \setminus X^t$. Set $t = t + 1$.
  - Repeat the process until $I^t = \emptyset$. Any remaining object s is left unassigned $\mu^{*-1}(s) = i_0$. The process results in the allocation μ^*.

Observe that this TTC algorithm does not need or use any exclusion right system by using only the preferences of agents and the priority structure to produce the final outcome μ^*. This is important, because this means that one can easily obtain the outcome μ^*. We will prove that μ^* is a proper exclusion right core allocation. To do so, we need to show that there exists a proper exclusion right system underlining μ^*. We use the TPC algorithm to achieve this. It is worth pointing out that we only need to know the existence of a proper exclusion right system underlying μ^* but do not need to have a concrete proper exclusion right system. This means that once we have established the existence of a proper exclusion right system, we can simply use the TTC algorithm to find the unique proper exclusion core allocation μ^* for any given relational economy with no need of using the TPC algorithm. This is very useful for any practical purpose.

The next result tells us that the TTC algorithm generates the same cycles as the TPC algorithm for the allocation μ^* and the threshold of an object is the agent to whom the object points in the TTC algorithm.

Lemma 2 *The TTC algorithm produces the same cycles as the TPC algorithm for the allocation μ^*.*

Theorem 3 *The TTC algorithm generates the unique proper exclusion right core allocation.*

3.3 Private information and incentive

A mechanism ϕ can be viewed as a function $\phi : \mathcal{P}^I \to \mathcal{A}$ that assigns every preference profile an allocation. We write a preference profile as $\succ = (\succ_i)_{i \in I}$ or $\succ = (\succ_i, \succ_{-i})$ for any $i \in I$ or $\succ = (\succ_C, \succ_{-C})$ for any coalition $C \subseteq I$. The top trading cycle algorithm that produces a proper exclusion right core allocation for every given relational economy with any preference profile is called a PEC TTC mechanism, simply denoted by $TTC_{pec}(\cdot)$.

We are interested in three fundamental properties: Pareto efficiency, no proper envy, and strategy-proofness, which are crucial and desirable to a good mechanism. A mechanism ϕ is **Pareto efficient** (PE) if the output allocation $\phi(\succ)$ is Pareto efficient under every preference profile $\succ \in \mathcal{P}^I$. We say that agent i properly envies agent j if $\mu(j) \succ_i \mu(i)$ and $i \blacktriangleright_\mu j$. A mechanism ϕ is **properly fair** if no agent properly envies any other agent at $\phi(\succ)(i)$ for all $\succ \in \mathcal{P}^I$. A mechanism ϕ is **strategy-proof** (SP) if $\phi(\succ)(i) \succeq_i \phi(\succ'_i, \succ_{-i})$ for all $i \in I$, all $\succ'_i$ and all $\succ_{-i}$. That is, a mechanism is strategy-proof if no agent can ever gain by unilaterally misrepresenting her preferences. A mechanism ϕ is **group strategy-proof** (GSP) if for every preference profile $\succ \in \mathcal{P}^I$, there do not exist a coalition $C \subseteq I$ and some preferences of the coalition $\succ'_C \in \mathcal{P}^C$ such that $\phi(\succ'_C, \succ_{-C})(i) \succeq_i \phi(\succ)(i)$ for all $i \in C$ and $\phi(\succ'_C, \succ_{-C})(j) \succ_j \phi(\succ)(j)$ for at least one $j \in C$. That is, a mechanism is group strategy-proof if no coalition of agents can ever gain by jointly acting dishonestly about their preferences. Clearly, a group strategy-proof mechanism must be strategy-proof. (Group) strategy-proofness is extremely important for a mechanism to be successful and ensures that it is optimal for every individual to act honestly. [5] Because preferences of

[5]See e.g., Hurwicz (1973); Roth (1982); Bird (1984); Ma (1994); Sönmez (1999); Pápai (2000); Ergin

every agent is her private information, one could reasonably expect agents to behave truthfully only if it is in their best interest of doing so.

Proposition 7 *The PEC TTC algorithm is group strategy-proof.*

The above proposition follows immediately from a well-known result due to Bird (1984) which improves the strategy-proof result of Roth (1982). Although their models consider exchange of private objects, their results can apply to the current model, because the incentive issue concerns about the possibility of manipulation by individuals on their preferences over objects not about who is endowed with which object.

Theorem 4 *A mechanism is properly fair, Pareto efficient, and strategy-proof if and only if it is the PEC TTC algorithm.*

4 An Extended Model

As an important extension of our previous one-to-one model, we consider a general school choice problem as a typical many-to-one example. A region has many (public) schools and many students. Each school has a capacity to admit multiple students and also has priorities over students. Every student has preferences over schools. The problem is described by a tuple $\langle I, S, Q, \succ, \rhd \rangle$, where $I = \{i_1, ..., i_{|I|}\}$ is a finite set of students and $S = \{s_1, ..., s_{|S|}\}$ is a finite set of schools. Each student $i \in I$ has a preference relation $\succ_i$ over $S \cup \{s_0\}$ where s_0 stands for a dummy position or school. Each school $s \in S$ has q_s capacities and a priority order $\rhd_s$ over students I. $Q = (q_s)_{s \in S}$ is the capacity vector, $\succ = (\succ_i)_{i \in I}$ is the preference profile, and $\rhd = (\rhd_s)_{s \in S}$ is the priority structure. As before we assume these preferences and priorities are strict, complete, and transitive.

For this model, an allocation is a function $\mu : I \to S \cup \{s_0\}$ such that $|\mu^{-1}(s)| \leq q_s$ for every school $s \in S$. This means that every student attends at most one school and every school admits students no more than its capacity. Let $\mathcal{A}$ be the set of all allocations for this model. The key difference of this model from the previous one is that every school $s \in S$ has a positive integer number q_s of positions and every student is indifferent to all positions in the same school, but has preferences only over schools.

(2002); Sun and Yang (2003); Kesten and Yazici (2012); Andersson and Svensson (2014); Pycia and Ünver (2017); Kamada and Kojima (2018); Sun and Yang (2021).

Here, a natural question is how to define an appropriate exclusion right system denoted by $(\blacktriangleright^p, \gg^p)$ and apply the results in the previous sections to this more general setting. To address the question, we consider an alternative but equivalent problem/model. We split every school $s \in S$ into q_s different seats $s^1, s^2, \cdots, s^{q_s}$. Let $\tilde{S} = \bigcup_{s\in S}\{s^1, \ldots, s^{q_s}\}$ be the set of all seats.

Every seat $s^\ell \in \tilde{S}$ has the same priorities over students as the school s has. Every student $i \in I$ has a strict preference relation $\tilde{\succ}_i$ over the seats in $\tilde{S}$ which is consistent with her original preference relation $\succ_i$. That is, for any two seats $s^\ell_m, s^k_n \in \tilde{S}$ of two different schools $s_m \neq s_n$, $s_m \succ_i s_n$ implies $s^\ell_m \tilde{\succ}_i s^k_n$; for any two seats $s^\ell_m, s^k_m \in \tilde{S}$ of the same school, agent i has a personal way to break ties. Clearly, the preference relation $\tilde{\succ}_i$ of every student i obtained in this way relies on the ordering of seats in $\tilde{S}$. Let $\mathcal{O}$ be the set of all such orderings. Let $\langle I, \tilde{S}, \tilde{\succ}, \tilde{\rhd}\rangle$ stand for the **alternative problem**.

For the alternative problem, we define an allocation as a function $\tilde{\mu} : I \to \tilde{S} \cup \{s_0\}$ such that $|\tilde{\mu}^{-1}(s^\ell)| \leq 1$ for every $s^\ell \in \tilde{S}$. Let $\tilde{\mathcal{A}}$ be the set of all such allocations. Clearly, every allocation $\tilde{\mu} \in \tilde{\mathcal{A}}$ corresponds to an allocation $f(\tilde{\mu}) \in \mathcal{A}$ in the original school choice problem. That is, for every student $i \in I$, $f(\tilde{\mu})(i) = s$ if $\tilde{\mu}(i) = s^\ell$ for some $\ell = 1, \ldots, q_s$ and $f(\tilde{\mu})(i) = s_0$ if $\tilde{\mu}(i) = s_0$. Note that two different allocations $\tilde{\mu}_1, \tilde{\mu}_2 \in \tilde{\mathcal{A}}$ may correspond to the same allocation $f(\tilde{\mu}_1) = f(\tilde{\mu}_2)$.

For any given allocation $\tilde{\mu} \in \tilde{\mathcal{A}}$, we can introduce a direct (exclusion right) scheme $\tilde{\blacktriangleright} = (\tilde{\blacktriangleright}_{\tilde{\mu}})_{\tilde{\mu}\in\tilde{\mathcal{A}}}$ and the corresponding derived (exclusion right) scheme $\tilde{\gg} = (\tilde{\gg}_{\tilde{\mu}})_{\tilde{\mu}\in\tilde{\mathcal{A}}}$. We can also define the core and the proper exclusion right core. For every given allocation $\tilde{\mu}$, we can use the TPC algorithm to obtain its threshold function $\theta^p_{\tilde{\mu}}$ and so obtain the corresponding proper exclusion right system $(\tilde{\blacktriangleright}^p, \tilde{\gg}^p)$.

Given the proper exclusion right system $(\tilde{\blacktriangleright}^p, \tilde{\gg}^p)$ generated by the TPC algorithm for the alternative problem, we can derive the exclusion right system $(\blacktriangleright^p, \gg^p)$ for the original model $\langle I, S, Q, \succ, \rhd\rangle$ as follows. For every allocation $\tilde{\mu} \in \tilde{\mathcal{A}}$ and its corresponding allocation $\mu = f(\tilde{\mu}) \in \mathcal{A}$, and all $i, j \in I$, (1) $i \blacktriangleright^p_\mu j$ if and only if $i \,\tilde{\blacktriangleright}^p_{\tilde{\mu}} j$; and (2) $i \gg^p_\mu j$ if and only if $i \,\tilde{\gg}^p_{\tilde{\mu}} j$.

The following lemma shows that the exclusion right system $(\blacktriangleright^p, \gg^p)$ derived from $(\tilde{\blacktriangleright}^p, \tilde{\gg}^p)$ is well-defined.

Lemma 3 *For any two allocations $\tilde{\mu}_1, \tilde{\mu}_2 \in \tilde{\mathcal{A}}$ such that $f(\tilde{\mu}_1) = f(\tilde{\mu}_2)$, they have the same* *exclusion right scheme:* $\tilde{\blacktriangleright}^p_{\tilde{\mu}_1} = \tilde{\blacktriangleright}^p_{\tilde{\mu}_2}$.

Now we have a proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\!\!\blacktriangleright^p)$ for the school choice problem. The core concept given by Definition 1 can be naturally adapted to the general model. Recall that given the derived exclusion right scheme $\blacktriangleright\!\!\!\!\blacktriangleright$, an allocation $\mu \in \mathcal{A}$ is blocked by a coalition $C \subseteq I$ if there exists another allocation $\nu \in \mathcal{A}$ such that $\nu(i) \succ_i \mu(i)$ for all $i \in C$ and $\mu(j) \succ_j \nu(j)$ implies that there is an agent $i \in C$ such that $i \blacktriangleright\!\!\!\!\blacktriangleright_\mu j$. The **core** is the set of allocations that cannot be blocked by any coalition.

Proposition 8 *Both the TPC-derived exclusion right system* $(\blacktriangleright^p, \blacktriangleright\!\!\!\!\blacktriangleright^p)$ *and the derived scheme* $\blacktriangleright\!\!\!\!\blacktriangleright^p$ *are proper for the original problem. If* $\tilde{\mu}$ *is the proper exclusion right core allocation of the alternative model, then* $f(\tilde{\mu})$ *is the proper exclusion right core allocation of the original model.*

Another important question is whether a different order of seats in the same school may lead to a different outcome for students. The following theorem says that no matter what ordering is chosen from $\mathcal{O}$, all proper exclusion right core allocations in the alternative problem will assign every student to the same school in S, although different students may be assigned to different schools in S. In other words, which school a student may get in is totally independent of the orderings. This is an important and desirable property.

Theorem 5 *Given the original model* $\langle I, S, Q, \succ, \rhd \rangle$*, every student will be assigned to the same school in* S *and different students may be assigned to different schools in* S *in all proper exclusion right core allocations of the alternative model* $\langle I, \tilde{S}, \tilde{\succ}, \tilde{\rhd} \rangle$ *no matter what ordering is taken from* $\mathcal{O}$.

Observe that unlike the model discussed in the previous sections, in the current school choice problem, because a school may have multiple seats and the priorities of students are given over every school not over its seats, this can create inconsistencies and ambiguities in the exclusion right scheme. To address this issue, we have used the following *coherent principle*: for every allocation $\mu \in \mathcal{A}$, (1) if $\mu(i) = s \in S$, then $k \rhd_s j \blacktriangleright_\mu i$ implies $k \blacktriangleright_\mu i$; and (2) if $\mu(i) = \mu(j) = s \in S$, then $\ell \rhd_s k \blacktriangleright_\mu i \rhd_s j$ implies $\ell \blacktriangleright_\mu i$ but does not imply $\ell \blacktriangleright_\mu j$ or $k \blacktriangleright_\mu j$.

The first part of the principle says that if student j has a right to exclude student i from school s and student k has a higher priority on school s than student j, then student k also has a right to exclude student j from school s; the second part says that if students i and j get in the same school s, student k has a right to exclude student i from school s, student l has a higher priority on school s than k, and student i has a higher priority on school s than j, then student l also has a right to exclude student i from school s but neither student l nor k is guaranteed to have a right to exclude student j from school s.

The next example shows that when the coherent principle is not observed, contradictory rights will emerge.

Example 3 *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2\}$ with $q_1 = 3$ and $q_2 = 1$. Students' preferences and schools' priorities are given by:*

$$\succ_{i_1} : s_2, s_1 \qquad \succ_{i_2}: s_2, s_1 \qquad \succ_{i_3}: s_1, s_2 \qquad \succ_{i_4}: s_1, s_2$$
$$\rhd_{s_1}: i_1, i_2, i_3, i_4 \qquad \rhd_{s_2} : i_4, i_1, i_2, i_3$$

There are two efficient allocations, $\mu_1 = \left(\begin{smallmatrix} i_1 & i_2 & i_3 & i_4 \\ s_2 & s_1 & s_1 & s_1 \end{smallmatrix}\right)$ and $\mu_2 = \left(\begin{smallmatrix} i_1 & i_2 & i_3 & i_4 \\ s_1 & s_2 & s_1 & s_1 \end{smallmatrix}\right)$. Allocation μ_1 is more reasonable since $i_1 \rhd_s i_2$ for all $s \in S$. At μ_1, we have $i_4 \blacktriangleright_{\mu_1} i_1$, $i_1 \blacktriangleright_{\mu_1} i_4$, $i_1 \blacktriangleright_{\mu_1} i_3$, $i_2 \blacktriangleright_{\mu_1} i_3$, and $i_3 \blacktriangleright_{\mu_1} i_3$. If $i_2 \rhd_{s_1} i_3 \blacktriangleright_{\mu_1} i_3 \rhd_{s_1} i_4$ should imply $i_2 \blacktriangleright_{\mu_1} i_4$, then we would have $i_2 \blacktriangleright_{\mu_1} i_4 \blacktriangleright_{\mu_1} i_1$. That is, i_2 would have an indirect right to exclude i_1 from s_2, resulting in μ_2. Obviously, i_1 should have a direct right to exclude i_2 from s_2 at μ_2. This creates a pair of contradictory exclusion rights.

In the following, we will show that if the coherent principle is not obeyed, the situation can get worse so the core becomes empty.

Definition 8 *A direct exclusion right scheme $\blacktriangleright$ is **excessively transferable** at allocation $\mu \in \mathcal{A}$ if $i \blacktriangleright_\mu j$ implies $i \blacktriangleright_\mu k$ for every $k \in \mu^{-1}(\mu(j))$ and $j \rhd_{\mu(j)} k$.*

The definition says that in such a scheme if student i can exclude student j from $\mu(j)$, then student i can also exclude student k who has a lower priority on $\mu(j)$ than j on $\mu(j)$.

The following example shows that an excessively transferable exclusion right scheme $\blacktriangleright$ can lead to an empty core.

Example 4 *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2\}$ with $q_1 = 3$ and $q_2 = 1$. Students' preferences and schools' priorities are given by:*

$$\succ_{i_1}: s_2, s_1 \qquad \succ_{i_2}: s_2, s_1 \qquad \succ_{i_3}: s_1, s_2 \qquad \succ_{i_4}: s_1$$
$$\triangleright_{s_1}: i_1, i_2, i_4, i_3 \qquad \triangleright_{s_2}: i_3, i_1, i_2, i_4$$

We now consider its alternative problem. Then we have $\tilde{S} = \{s_1^1, s_1^2, s_1^3, s_2^1\}$ and the preferences of every student and priorities of every school.

$$\tilde{\succ}_{i_1}: s_2^1, s_1^1, s_1^2, s_1^3 \qquad \tilde{\succ}_{i_2}: s_2^1, s_1^1, s_1^2, s_1^3 \qquad \tilde{\succ}_{i_3}: s_1^1, s_1^2, s_1^3, s_2^1 \qquad \tilde{\succ}_{i_4}: s_1^1, s_1^2, s_1^3$$
$$\tilde{\triangleright}_{s_1^1}: i_1, i_2, i_4, i_3 \qquad \tilde{\triangleright}_{s_1^2}: i_1, i_2, i_4, i_3 \qquad \tilde{\triangleright}_{s_1^3}: i_1, i_2, i_4, i_3 \qquad \tilde{\triangleright}_{s_2^1}: i_3, i_1, i_2, i_4$$

Since all inefficient allocations are not in the core, we only need to consider efficient allocations μ_1 and μ_2 as shown in Table 3. Their corresponding allocations in the alternative model are $\tilde{\mu}_1$ and $\tilde{\mu}_2$, respectively.

Table 3: Efficient allocations and exclusion right system

	s_1	s_2		s_1^1	s_1^2	s_1^3	s_2^1	The direct exclusion right system $\tilde{\blacktriangleright}^p$
μ_1	i_2, i_3, i_4	i_1	$\tilde{\mu}_1$	i_2	i_4	i_3	i_1	$i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_2$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_3$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_2$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$, $i_3 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_1$, $i_4 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$
μ_2	i_1, i_3, i_4	i_2	$\tilde{\mu}_2$	i_1	i_4	i_3	i_2	$i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_1$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_3$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_3$, $i_3 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_2$, $i_4 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$

	s_1	s_2		s_1^1	s_1^2	s_1^3	s_2^1	The direct exclusion right system $\tilde{\blacktriangleright}'$
μ_1	i_2, i_3, i_4	i_1	$\tilde{\mu}_1$	i_2	i_4	i_3	i_1	$i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_2$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_3$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_2$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_3$, $i_3 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_1$, $i_4 \tilde{\blacktriangleright}_{\tilde{\mu}_1} i_4$
μ_2	i_1, i_3, i_4	i_2	$\tilde{\mu}_2$	i_1	i_4	i_3	i_2	$i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_1$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$, $i_1 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_3$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$, $i_2 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_3$, $i_3 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_2$, $i_4 \tilde{\blacktriangleright}_{\tilde{\mu}_2} i_4$

The proper direct exclusion right scheme $\tilde{\blacktriangleright}^p$ in Table 3 is produced by the TPC algorithm. We follow the coherent principle here. We can see that in the system $\tilde{\blacktriangleright}^p_{\tilde{\mu}_1}$, student i_2 directly exclude student i_4 from her occupied seat s_1^2, but cannot directly exclude i_3 from seat s_1^3, even though i_3 has a lower priority than i_4 at both seats s_1^2 and s_1^3.

Now consider a direct exclusion right scheme $\tilde{\blacktriangleright}'$, in which i_2 can directly exclude both i_4 and i_3 from seats s_1^2 and s_1^3, respectively, as shown in Table 3. Note that the derived exclusion right scheme $\tilde{\blacktriangleright\!\!\triangleright}'$ creates contradictory rights, i.e., $i_2 \tilde{\blacktriangleright\!\!\triangleright}_{\tilde{\mu}_1} i_1 \tilde{\blacktriangleright\!\!\triangleright}_{\tilde{\mu}_2} i_2$. Here the coherent principle is not observed, because the scheme $\tilde{\blacktriangleright\!\!\triangleright}'$ is excessively transferable at μ_1. We will show that the core is empty. Since every allocation in the core is efficient, we only need to consider efficient allocations μ_1 and μ_2, and their corresponding allocations $\tilde{\mu}_1$ and $\tilde{\mu}_2$. At allocation $\tilde{\mu}_1$, student i_2 can indirectly exclude i_1 from seat s_2^1 (i.e. $i_2 \tilde{\blacktriangleright\!\!\triangleright}_{\tilde{\mu}_1} i_1$).

While, at allocation $\tilde{\mu}_2$, student i_1 can indirectly exclude i_2 from seat s_2^1 (i.e. $i_1 \tilde{\blacktriangleright\!\!\!\blacktriangleright}_{\tilde{\mu}_2} i_2$). Therefore, the core is empty.

5 A Comparison with Balbuzanov and Kotowski (2019) and Reny (2022)

As mentioned earlier, our study is closely related to Balbuzanov and Kotowski (2019) and Reny (2022). In this section we compare our solution with theirs in detail. For the convenience of comparison, we summarize the key features of our proper exclusion right core. This solution is built on the unique exclusion right system and possesses the properties of efficiency, existence, uniqueness, weak and proper fairness, competitiveness, and incentive compatibility. The proper exclusion right system identifies a proper range of exclusion rights which do not clash with each other (i.e. contradictory rights are eliminated) and at the same time attain the maximum that individuals can possibly enjoy. This system also gives an arguably plausible explanation of the root causes of the tragedies of the commons and the anticommons.

5.1 The solutions of Balbuzanov and Kotowski (2019)

Balbuzanov and Kotowski (2019) introduce three different exclusion cores based on endowments of individuals, their extended endowments and given priorities. Our Definition 1 of core in Section 2 is close to the strong exclusion core in their Section 4 as stated below. To be consistent, we replace their set H of houses by S.

Definition 9 *A **weak conditional endowment system** at allocation μ is $\omega_\mu^w : I \to 2^S$ such that, for every $i \subseteq I$, $s \in \omega_\mu^w(i)$ if and only if $i \trianglerighteq_s \mu^{-1}(s)$. An allocation is in the **strong exclusion core** if it is not indirectly blocked by any coalition given ω_μ^w.*

We will show by Example 1 in Section 2 that the *strong exclusion core is empty* but the proper exclusion right core is not empty and contains a single solution. We use ($\blacktriangleright^w$, $\blacktriangleright\!\!\!\blacktriangleright^w$) to represent the corresponding exclusion right system of the weak conditional endowment system. The proper exclusion right system is denoted by ($\blacktriangleright^p$, $\blacktriangleright\!\!\!\blacktriangleright^p$).

In Example 1, there are 13 potential allocations. Since inefficient allocations are not in the strong exclusion core or the proper exclusion right core, we only need to consider efficient allocations μ_1, μ_2, μ_3 and μ_5. The exclusion right system $(\blacktriangleright, \blacktriangleright\!\!\blacktriangleright)$ as shown in Table 1 corresponds to the weak conditional endowment system $(\blacktriangleright^w, \blacktriangleright\!\!\blacktriangleright^w)$. The strong exclusion core is empty, as allocation μ_1 is indirectly blocked by coalition $\{i_3\}$ because $\mu_1(i_2) \succ_{i_3} s_0$ and $i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_1} i_1$, allocation μ_2 is indirectly blocked by coalition $\{i_2\}$ because $\mu_1(i_3) \succ_{i_2} s_0$ and $i_2 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_3$, allocation μ_3 is indirectly blocked by coalition $\{i_3\}$, and allocation μ_5 is indirectly blocked by coalition $\{i_1\}$. The weak conditional endowment system $(\blacktriangleright^w, \blacktriangleright\!\!\blacktriangleright^w)$ has contradictory rights because there exist two allocations μ_2 and μ_5 such that $\mu_2(i_3) = \mu_5(i_3) = s_2$, $\mu_2(i_1) = \mu_5(i_2) = s_1$, and $i_1 \blacktriangleright_{\mu_5} i_2 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_1$. The proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\blacktriangleright^p)$ for Example 1 is shown in Table 4. Given the proper system $(\blacktriangleright^p, \blacktriangleright\!\!\blacktriangleright^p)$, allocation μ_2 is the unique proper exclusion right core outcome.

Balbuzanov and Kotowski (2019, p.1676) also propose strong conditional endowment system, weak exclusion core, unconditional endowment system and unconditional exclusion core. We use Example 5 below to show that both weak and unconditional exclusion cores may contain weakly unfair outcomes but the proper exclusion right core eliminates those undesirable outcomes. Here $(\blacktriangleright^v, \blacktriangleright\!\!\blacktriangleright^v)$ and $(\blacktriangleright^u, \blacktriangleright\!\!\blacktriangleright^u)$ stand for the corresponding exclusion right systems of the strong conditional endowment system and unconditional endowment system, respectively. Note that Example 5 differs from Example 1 only in i_3's preferences.

Example 5 *Let $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2\}$. The preferences $\succ$ and priority structure $\rhd$ are given by:*

$$\rhd_{s_1}: i_3, i_1, i_2 \quad \rhd_{s_2}: i_1, i_2, i_3 \qquad \succ_{i_1}: s_1, s_2, s_0 \quad \succ_{i_2}: s_1, s_2, s_0 \quad \succ_{i_3}: s_0, s_2, s_1$$

As both Examples 1 and 5 have the same priority structure, they share the same proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\blacktriangleright^p)$ in Table 4. This table also shows the exclusion right system $(\blacktriangleright^v, \blacktriangleright\!\!\blacktriangleright^v)$ and the exclusion right system $(\blacktriangleright^u, \blacktriangleright\!\!\blacktriangleright^u)$. In the table, $i_1 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_1$ comes from $i_1 \blacktriangleright_{\mu_2} i_3 \blacktriangleright_{\mu_2} i_1$, $i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_2} i_3$ from $i_3 \blacktriangleright_{\mu_2} i_2 \blacktriangleright_{\mu_2} i_3$, $i_2 \blacktriangleright\!\!\blacktriangleright_{\mu_5} i_2$ from $i_2 \blacktriangleright_{\mu_5} i_3 \blacktriangleright_{\mu_5} i_2$, and $i_3 \blacktriangleright\!\!\blacktriangleright_{\mu_5} i_3$ from $i_3 \blacktriangleright_{\mu_5} i_2 \blacktriangleright_{\mu_5} i_3$. We can easily verify that the weak and unconditional exclusion cores coincide and equal $\{\mu_1, \mu_3\}$. However, allocation μ_3 is not weakly fair because agent i_1 prefers $\mu_3(i_2) = s_1$ to his assignment $\mu_3(i_1) = s_2$ and i_1 has a higher priority than i_2 at both s_1 and s_2. Following Dur and Morrill (2018), we can also show that μ_3 cannot be supported by competitive prices. Assume that $p_{s_1} > 0$, $p_{s_2} > 0$ and $p_{s_0} = 0$ are the equilibrium prices of the houses. Let $p_i(s_j)$ be the value of agent i on house s_j. To respect and reflect the given priorities, we can set $p_{i_3}(s_1) \geq p_{i_1}(s_1) \geq p_{i_2}(s_1)$ and $p_{i_1}(s_2) > p_{i_2}(s_2) = p_{i_3}(s_2) = 0$. This implies that i_1's income of $\max\{p_{i_1}(s_1), p_{i_1}(s_2)\}$ is no less than i_2's income of $\max\{p_{i_2}(s_1), p_{i_2}(s_2)\}$. In equilibrium, every agent's income must be equal to her spending. Clearly, s_1 is overdemanded by i_1 and i_2 at μ_3, yielding a contradiction.

The proper exclusion right core equals $\{\mu_1\}$ and eliminates the undesirable allocation μ_3. Here we can see that the exclusion right system ($\blacktriangleright^v$, $\blacktriangleright\!\!\!\blacktriangleright^v$) of the strong conditional endowment system has unnecessarily eliminated some exclusion rights so that it has generated some undesirable outcomes in the weak exclusion core. Take allocation μ_3 as an illustration. At the allocation, agent i_1 does not have an exclusion right to i_2 in the system ($\blacktriangleright^v$, $\blacktriangleright\!\!\!\blacktriangleright^v$), but i_1 could actually have an exclusion right to i_2 like the one in the proper system ($\blacktriangleright^p$, $\blacktriangleright\!\!\!\blacktriangleright^p$), which can still guarantee a nonempty core.

To ensure a nonempty exclusion core, Balbuzanov and Kotowski (2019) provide two methods to deal with problematic cycles rooted in the priority structure. Their first method is to require the priority structure to be acyclic. They prove that for their relational economy with an acyclic priority structure (see also Ergin (2002)), their strong and weak exclusion cores are not empty and coincide. Notice that our model does not require an acyclic priority structure.

Their second method is to impose conditions directly on the endowment system so that potential exclusion rights which are susceptible to problematic cycles will be excluded. To understand their second method, let us look at allocations μ_2, μ_5 and μ_{12} in Table 4. In the system ($\blacktriangleright^v$, $\blacktriangleright\!\!\!\blacktriangleright^v$), the direct exclusion rights $i_2 \blacktriangleright_{\mu_2} i_3$, $i_2 \blacktriangleright_{\mu_5} i_3$, and $i_2 \blacktriangleright_{\mu_{12}} i_3$ have been excluded because they are seen as vulnerable to the problematic cycle $i_1 \rhd_{s_2} i_2 \rhd_{s_2} i_3 \rhd_{s_1} i_1$. In contrast, our proper exclusion right system ($\blacktriangleright^p$, $\blacktriangleright\!\!\!\blacktriangleright^p$) just removes $i_2 \blacktriangleright_{\mu_2} i_3$ but recognizes the other two. The problematic cycle cannot be credibly formed at μ_5 or μ_{12}, because i_1 is not the occupant of s_1 but s_0, so i_3 has no basis to exclude i_1 from s_1. However, the problematic cycle can be credibly formed at μ_2, as i_1 occupies s_1 and i_3 indeed can exclude i_1 from s_1. Therefore it is natural and sensible to recognize both exclusion rights $i_2 \blacktriangleright_{\mu_5} i_3$ and $i_2 \blacktriangleright_{\mu_{12}} i_3$ but remove $i_2 \blacktriangleright_{\mu_2} i_3$. Balbuzanov and Kotowski's second method tries to limit the scope of exclusion rights but has a danger of overkill.

Table 4: The exclusion right systems

μ	i_1 i_2 i_3	$(\blacktriangleright^v, \blacktriangleright\!\!\!\triangleright^v)$	$(\blacktriangleright^u, \blacktriangleright\!\!\!\triangleright^u)$	$(\blacktriangleright^p, \blacktriangleright\!\!\!\triangleright^p)$
μ_1	s_1 s_2 s_0	$i_3 \blacktriangleright_{\mu_1} i_1, i_1 \blacktriangleright_{\mu_1} i_1$ $i_1 \blacktriangleright_{\mu_1} i_2, i_2 \blacktriangleright_{\mu_1} i_2, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_1} i_2$	$i_3 \blacktriangleright_{\mu_1} i_1$ $i_1 \blacktriangleright_{\mu_1} i_2, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_1} i_2$	$i_3 \blacktriangleright_{\mu_1} i_1, i_1 \blacktriangleright_{\mu_1} i_1,$ $i_1 \blacktriangleright_{\mu_1} i_2, i_2 \blacktriangleright_{\mu_1} i_2, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_1} i_2$
μ_2	s_1 s_0 s_2	$i_3 \blacktriangleright_{\mu_2} i_1, i_1 \blacktriangleright_{\mu_2} i_1$ $i_1 \blacktriangleright_{\mu_2} i_3, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_2} i_3$	$i_3 \blacktriangleright_{\mu_2} i_1, i_1 \blacktriangleright\!\!\!\triangleright_{\mu_2} i_1$ $i_1 \blacktriangleright_{\mu_2} i_3, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_2} i_3$	$i_3 \blacktriangleright_{\mu_2} i_1, i_1 \blacktriangleright\!\!\!\triangleright_{\mu_2} i_1$ $i_1 \blacktriangleright_{\mu_2} i_3, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_2} i_3$
μ_3	s_2 s_1 s_0	$i_1 \blacktriangleright_{\mu_3} i_1, i_3 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$ $i_3 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$	$i_1 \blacktriangleright_{\mu_3} i_1, i_3 \blacktriangleright_{\mu_3} i_2$ $i_3 \blacktriangleright_{\mu_3} i_2$	$i_1 \blacktriangleright_{\mu_3} i_1, i_3 \blacktriangleright_{\mu_3} i_2, i_1 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$ $i_3 \blacktriangleright_{\mu_3} i_2, i_1 \blacktriangleright_{\mu_3} i_2, i_2 \blacktriangleright_{\mu_3} i_2$
μ_4	s_2 s_0 s_1	$i_1 \blacktriangleright_{\mu_4} i_1, i_3 \blacktriangleright_{\mu_4} i_3$	$i_1 \blacktriangleright_{\mu_4} i_1, i_3 \blacktriangleright_{\mu_4} i_3$	$i_1 \blacktriangleright_{\mu_4} i_1, i_3 \blacktriangleright_{\mu_4} i_3$
μ_5	s_0 s_1 s_2	$i_3 \blacktriangleright_{\mu_5} i_2, i_1 \blacktriangleright\!\!\!\triangleright_{\mu_5} i_2$ $i_1 \blacktriangleright_{\mu_5} i_3, i_3 \blacktriangleright_{\mu_5} i_3$	$i_3 \blacktriangleright_{\mu_5} i_2, i_1 \blacktriangleright\!\!\!\triangleright_{\mu_5} i_2$ $i_1 \blacktriangleright_{\mu_5} i_3$	$i_3 \blacktriangleright_{\mu_5} i_2, i_1 \blacktriangleright\!\!\!\triangleright_{\mu_5} i_2, i_2 \blacktriangleright\!\!\!\triangleright_{\mu_5} i_2$ $i_1 \blacktriangleright_{\mu_5} i_3, i_2 \blacktriangleright_{\mu_5} i_3, i_3 \blacktriangleright\!\!\!\triangleright_{\mu_5} i_3$
μ_6	s_0 s_2 s_1	$i_1 \blacktriangleright_{\mu_6} i_2, i_2 \blacktriangleright_{\mu_6} i_2, i_3 \blacktriangleright_{\mu_6} i_3$	$i_1 \blacktriangleright_{\mu_6} i_2, i_3 \blacktriangleright_{\mu_6} i_3$	$i_1 \blacktriangleright_{\mu_6} i_2, i_2 \blacktriangleright_{\mu_6} i_2, i_3 \blacktriangleright_{\mu_6} i_3$
μ_7	s_1 s_0 s_0	$i_3 \blacktriangleright_{\mu_7} i_1, i_1 \blacktriangleright_{\mu_7} i_1$	$i_3 \blacktriangleright_{\mu_7} i_1$	$i_3 \blacktriangleright_{\mu_7} i_1, i_1 \blacktriangleright_{\mu_7} i_1$
μ_8	s_2 s_0 s_0	$i_1 \blacktriangleright_{\mu_8} i_1$	$i_1 \blacktriangleright_{\mu_8} i_1$	$i_1 \blacktriangleright_{\mu_8} i_1$
μ_9	s_0 s_1 s_0	$i_3 \blacktriangleright_{\mu_9} i_2, i_2 \blacktriangleright_{\mu_9} i_2$	$i_3 \blacktriangleright_{\mu_9} i_2$	$i_3 \blacktriangleright_{\mu_9} i_2, i_1 \blacktriangleright_{\mu_9} i_2, i_2 \blacktriangleright_{\mu_9} i_2$
μ_{10}	s_0 s_2 s_0	$i_1 \blacktriangleright_{\mu_{10}} i_2, i_2 \blacktriangleright_{\mu_{10}} i_2$	$i_1 \blacktriangleright_{\mu_{10}} i_2$	$i_1 \blacktriangleright_{\mu_{10}} i_2, i_2 \blacktriangleright_{\mu_{10}} i_2$
μ_{11}	s_0 s_0 s_1	$i_3 \blacktriangleright_{\mu_{11}} i_3$	$i_3 \blacktriangleright_{\mu_{11}} i_3$	$i_3 \blacktriangleright_{\mu_{11}} i_3$
μ_{12}	s_0 s_0 s_2	$i_1 \blacktriangleright_{\mu_{12}} i_3, i_3 \blacktriangleright_{\mu_{12}} i_3$	$i_1 \blacktriangleright_{\mu_{12}} i_3$	$i_1 \blacktriangleright_{\mu_{12}} i_3, i_2 \blacktriangleright_{\mu_{12}} i_3, i_3 \blacktriangleright_{\mu_{12}} i_3$
μ_{13}	s_0 s_0 s_0	$\emptyset$	$\emptyset$	$\emptyset$

5.2 The solution of Reny (2022)

Reny (2022) proposes the solution concept of priority-efficiency for the school choice problem. To be consistent, an allocation in this section will be called a matching. Given a problem $\langle I, S, \succ, \rhd \rangle$, we say that student $i \in I \cup \{i_0\}$ violates student j's priority ($j \neq i$) at a matching $\mu \in \mathcal{A}$ if $j \rhd_{\mu(i)} i$ and $\mu(i) \succ_j \mu(j)$, which will be denoted by (j, i). Recall i_0 is the virtual agent. We call (j, i) a priority violation pair, i a p-violator and j a p-victim. A priority violation pair (j, i) at a matching μ is *R-tolerable* if improving the p-victim j via any new matching ν would generate a new priority violation pair (j', i') in which the new p-victim j' prefers $\mu(j')$ to $\nu(j')$. A matching is *priority-neutral* if every priority violation pair is tolerable. Note that this definition of priority-neutrality is an alternative but equivalent form of Reny's original one. A matching is *priority-efficient* if it is both Pareto efficient and priority neutral. Reny (2022) proves that a priority-neutral matching respects the right to relief and equal priority rights.

In order to compare Reny's solution with ours, we say that a priority violation pair (j, i) at a matching μ is R^{PER}-*tolerable* if $j \not\mathrel{\blacktriangleright\!\!>}^{p}_{\mu} i$ (i.e. j does not have any exclusion right to i) under the proper exclusion right system $(\blacktriangleright^{p}, \mathrel{\blacktriangleright\!\!>}^{p})$. We can easily prove that every priority violation pair in the proper exclusion right core is R^{PER}-tolerable. However, unlike the priority-efficient matching, Pareto-efficiency together with R^{PER}-tolerability is insufficient to describe the proper exclusion right core. In general, R^{PER}-tolerability is quite different from R-tolerability, although they can be identical sometimes.

We use the following example to compare Reny's solution and ours.

Example 6 *There are three students* $\{i_1, i_2, i_3\}$ *and three schools* $\{s_1, s_2, s_3\}$. *The preferences of agents and priority structure are given in Table 5.*

Table 5: Preferences of agents and priority structure.

i_1	i_2	i_3	s_1	s_2	s_3
s_1^*	s_2^*	s_2	i_3	i_1	i_1
s_2	s_1	s_3^*	i_2	i_3	i_3^*
s_3	s_3	s_1	i_1^*	i_2^*	i_2

Observe that this example has two Pareto efficient allocations: the circled matching μ° and the asterisked matching μ^* shown in Table 5. Table 6 shows all priority violation pairs and exclusion rights at μ° and μ^* of the proper exclusion right system.

Table 6: Priority violations at μ^* and exclusion rights at μ°

μ	i_1	i_2	i_3	Priority violation	R-tolerable	R^{PEC}-tolerable	Exclusion rights
μ°	s_1	s_3	s_2	(i_2, i_1)	$\emptyset$	(i_2, i_1)	$i_3 \blacktriangleright_{\mu^\circ} i_1, i_1 \blacktriangleright_{\mu^\circ} i_3, i_1 \blacktriangleright_{\mu^\circ} i_2, i_3 \blacktriangleright_{\mu^\circ} i_2,$ $i_2 \blacktriangleright_{\mu^\circ} i_2, i_3 \mathrel{\blacktriangleright\!\!>}_{\mu^\circ} i_3, i_1 \mathrel{\blacktriangleright\!\!>}_{\mu^\circ} i_1$
μ^*	s_1	s_2	s_3	(i_3, i_2)	(i_3, i_2)	$\emptyset$	$i_3 \blacktriangleright_{\mu^*} i_1, i_1 \blacktriangleright_{\mu^*} i_2, i_3 \blacktriangleright_{\mu^*} i_2, i_1 \blacktriangleright_{\mu^*} i_3,$ $i_1 \mathrel{\blacktriangleright\!\!>}_{\mu^*} i_1, i_3 \mathrel{\blacktriangleright\!\!>}_{\mu^*} i_3$

We first prove that μ^* is priority-efficient but not in the proper exclusion right core. At μ^*, there is one priority violation pair (i_3, i_2). We will show that (i_3, i_2) is R-tolerable. To make the only p-victim i_3 better off, i_3 should be assigned to s_2. We show that any matching μ' satisfying $\mu'(i_3) = s_2$ will make some student j worse off and also generate a new priority violation pair (j, k) for some student k. We only need to consider two matchings μ° and μ' where $\mu'(i_3) = s_2, \mu'(i_1) = s_3$ and $\mu'(i_2) = s_1$. At μ°, student i_2 prefers $\mu^*(i_2) = s_2$ to $\mu^\circ(i_2) = s_3$ and there is a new priority violation pair (i_2, i_1). Similarly, at μ', student i_1 prefers $\mu^*(i_1) = s_1$ to $\mu'(i_1) = s_3$ and there is a new priority violation pair (i_1, i_3). Note that the priority violation pair (i_3, i_2) is not R^{PEC}-tolerable, because $i_3 \blacktriangleright_{\mu^*} i_2$ under the proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\blacktriangleright^p)$. Therefore, μ^* is priority-efficient but not in the proper exclusion right core.

Next, we look at μ°. There is one priority violation pair (i_2, i_1). To show the pair (i_2, i_1) is not R-tolerable, consider matching μ' given by $\mu'(i_1) = s_2, \mu'(i_2) = s_1$ and $\mu'(i_3) = s_3$. Improving the p-victim i_2 by assigning s_1 to i_2 via μ' will not violate any student's priority. The pair (i_2, i_1) is R^{PEC}-tolerable, because i_2 has no exclusion right to i_1, i.e. $i_2 \not\blacktriangleright\!\!\blacktriangleright_{\mu^\circ} i_1$ under the proper exclusion right system $(\blacktriangleright^p, \blacktriangleright\!\!\blacktriangleright^p)$. This shows that μ° is a proper exclusion right core matching but not priority-efficient.

We now discuss four other differences. First, observe that Reny's solution μ^* is not weakly fair, because i_3 has a higher priority than i_2 at every school and prefers i_2's assignment $\mu^*(i_2) = s_2$ to hers $\mu^*(i_3) = s_3$. Second, as to be shown in Section 3.3, the mechanism for finding our proper exclusion right core matching can induce students to act honestly but for Reny's solution incentive-compatibility cannot be achieved. To see this, for Reny's solution i_3 has an incentive to misreport her preference to obtain a better assignment. For example, if i_3 reports the preferences $P'_{i_3} : s_2, s_1, s_3$ and i_1, i_2 maintain their true preferences, then both priority-efficient matching and proper exclusion right core matching are μ° and i_3 gets s_2 and is better off. Reny uses the algorithm introduced by Kesten (2010) and improved by Tang and Yu (2014) to find the priority-efficient matching and it is known that the algorithm is not strategy-proof.

Third, we will prove that Reny's solution μ^* cannot be supported by any competitive prices. Assume that $p_{s_1} > 0$, $p_{s_2} > 0$, and $p_{s_3} > 0$ are the competitive prices of s_1, s_2, and s_3, respectively, to support μ^*. In equilibrium, i_1, i_2 and i_3 must have their incomes p_{i_1}, p_{i_2} and p_{i_3} equal to their respective spending p_{s_1}, p_{s_2} and p_{s_3}. For every position s_j, let $p_i(s_j)$ be the value of agent i. To respect and reflect the given priorities, we can set $p_{i_3}(s_1) \geq p_{i_2}(s_1) \geq p_{i_1}(s_1)$ for s_1, $p_{i_1}(s_2) \geq p_{i_3}(s_2) \geq p_{i_2}(s_2)$ for s_2, and $p_{i_1}(s_3) \geq p_{i_3}(s_3) > p_{i_2}(s_3) = 0$ for s_3. Then the income of every agent i equals $p_i = \max\{p_i(s_1), p_i(s_2), p_i(s_3)\}$. It follows that we have $p_{i_3} \geq p_{i_2}$ so s_2 is overdemanded by i_3 and i_2, yielding a contradiction.

Finally, Reny's solution respects the right to relief and equal priority rights. These rights use priority violations and depend on both the given priorities and agents' preferences. They are basically different from our proper exclusion rights. Our proper exclusion right core respects the proper exclusion rights. These rights are solely derived or inherited from the given priorities and are independent of agents' preferences.

6 Conclusion

We have studied the problem of how to allocate multiple indivisible items such as positions and houses to several individuals in a competitive, efficient, fair, and incentive compatible way. The items are typically not private and may belong to a community, an organization, or the public. There is no medium of exchange such as money. Every individual demands at most one item and has personal preferences over the items. The right of using these items relies on exogenously given priorities. But the rights and preferences of individuals are often competing. We have introduced the proper exclusion right system which identifies a proper range of exclusion rights, and shown its existence and uniqueness. The key contribution of the paper is the development of proper exclusion right core. This new core always exists and contains exactly one solution, which is efficient, properly and weakly fair, can be supported by competitive prices and easily found by the TTC mechanism in a group strategy-proof way. We have also established that a mechanism is efficient, properly fair and strategy-proof if and only if it is the TTC mechanism that produces the unique proper exclusion right core outcome. Furthermore, we have considered an extension of the model and obtained several results.

We have compared the proper exclusion right core with the solutions of Balbuzanov and Kotowski (2019) and Reny (2022) in detail. It is also worth comparing our work with two early important related studies. Ergin (2002) has shown in his main result the equivalence between acyclicity of the priority structure, Pareto efficiency, group strategy-proofness, and consistency. Our proper core solution shares Pareto efficiency and group strategy-proofness with his but also has markedly different properties such as weak and proper fairness, competitiveness and is conceptually different from his.

Our solution does not need acyclicity on priorities. Hylland and Zeckhauser (1979) have considered a similar but different model. Their solution is based on lotteries so it offers probability distributions of positions among individuals and is conceptually totally different from ours. They have suggested a procedure which does not always guarantee to find a solution. In contrast, the TTC mechanism used in the current article can easily find the unique proper exclusion core outcome and prevent any manipulation or collusion by any individual or any group of individuals.

The salient feature of our proper exclusion right core is its constant existence and its unique solution, sharply contrasting with many existing core concepts which either may contain many solutions and some of them can be undesirable or contains no solution at all. The unique proper exclusion right core allocation can be easily found and implemented and has also demonstrated its strong explanatory, predictive power and merits. As a byproduct, our results have also shed new light on the tragedies of both the commons and the anticommons. From the examination of the two tragedies, we have come to understand that identifying proper exclusion rights has played a crucial role in solving our current problem. We hope our new solution could someday find its way into practical usage and our analysis can be applied to other exchange and allocation problems of particularly non-private resources.

A Appendix: Proofs

Proof of Proposition 1: Suppose on the contrary that the derived scheme $\blacktriangleright$ were not self-consistent. Then there would exist two agents $\{i, j\}$ and two allocations $\mu, \nu \in \mathcal{A}$, such that $s = \mu(i) = \nu(j)$ and $\mu(k) = \nu(k)$ for all other agents $k \in I \setminus \{i, j\}$, and $i \blacktriangleright_\nu j \blacktriangleright_\mu i$.

We show that the exclusion core is empty under the following preference profile $\succ = (\succ_k)_{k \in I}$. For each agent $k \in I$, define

$$\succ_k = \begin{cases} \mu(k), s_0, & \text{if } k \in I \setminus \{i, j\} \\ s, s_0, & \text{if } k \in \{i, j\} \end{cases} \tag{A.1}$$

Given the preference profile, any allocation μ' in which $\mu'(k) = s_0$ for some $k \in I \setminus \{i, j\}$ is blocked by coalition $\{k\}$, because no one else prefers $\mu(k)$ and would not be hurt by assigning $\mu(k)$ to k. Now, the remaining alternatives are μ and ν, where $\mu(j) = \nu(i) = s_0$. Allocation μ is blocked by coalition $\{j\}$ since $j \blacktriangleright\!\!\!\blacktriangleright_\mu i$. Similarly, allocation ν is blocked by coalition $\{i\}$ since $i \blacktriangleright\!\!\!\blacktriangleright_\nu j$. $\square$

Proof of Lemma 1: Let $\blacktriangleright\!\!\!\blacktriangleright$ be a self-consistent exclusion right scheme. Assume on the contrary that there existed $\mu \in \mathcal{A}$, $i \in I$ and $\mu(i) \in S$ such that $i \not\blacktriangleright\!\!\!\blacktriangleright_\mu i$. Let $\blacktriangleright$ be the original direct exclusion right scheme of the scheme $\blacktriangleright\!\!\!\blacktriangleright$. Define $\blacktriangleright'$ by adding a relation $i \blacktriangleright'_\mu i$ to $\blacktriangleright$. Let $\blacktriangleright\!\!\!\blacktriangleright'$ be the derived scheme from the scheme $\blacktriangleright'$.

We first show that $\blacktriangleright\!\!\!\blacktriangleright'$ would be larger than $\blacktriangleright\!\!\!\blacktriangleright$. We have $i \blacktriangleright\!\!\!\blacktriangleright'_\mu i$ implied by $i \blacktriangleright'_\mu i$ and $i \not\blacktriangleright\!\!\!\blacktriangleright_\mu i$ by assumption. We now prove that $j \blacktriangleright\!\!\!\blacktriangleright_\nu k \Leftrightarrow j \blacktriangleright\!\!\!\blacktriangleright'_\nu k$ holds for any of the remaining cases that $\nu \in \mathcal{A}$ and $j, k \in I$ such that either $\nu \neq \mu$, or $j \neq i$, or $k \neq i$. If $j \blacktriangleright\!\!\!\blacktriangleright_\nu k$, then there exists a nonempty sequence of agents such that $j \blacktriangleright_\nu j_1 \blacktriangleright_\nu \cdots \blacktriangleright_\nu j_L \blacktriangleright_\nu k$. It is convenient to assume that the agents in the sequence are different. Otherwise, if an agent takes two positions in the sequence (i.e., $j_\ell = j_{\ell+m}$), we can shorten the sequence by removing the cycle (i.e., let $j \blacktriangleright_\nu j_1 \blacktriangleright_\nu \cdots \blacktriangleright_\nu j_\ell \blacktriangleright_\nu j_{\ell+m+1} \blacktriangleright_\nu \cdots \blacktriangleright_\nu j_L \blacktriangleright_\nu j$ be the sequence). By the definition of $\blacktriangleright'$, we also have $j \blacktriangleright'_\nu j_1 \blacktriangleright'_\nu \cdots \blacktriangleright'_\nu j_L \blacktriangleright'_\nu k$, so $j \blacktriangleright'_\nu k$. The reverse is also true. So, the first requirement is satisfied.

We now show that $\blacktriangleright\!\!\!\blacktriangleright'$ is also self-consistent. Suppose on the contrary that $\blacktriangleright\!\!\!\blacktriangleright'$ had contradictory rights. Then there would exist two different agents $\{j_1, j_2\}$ and two allocations $\mu_1, \mu_2 \in \mathcal{A}$ such that $h = \mu_1(j_1) = \mu_2(j_2)$, $\mu_1(k) = \mu_2(k)$ for all other agents $k \in I \setminus \{j_1, j_2\}$, and $j_2 \blacktriangleright\!\!\!\blacktriangleright'_{\mu_1} j_1 \blacktriangleright\!\!\!\blacktriangleright'_{\mu_2} j_2$. Then we have $j_2 \blacktriangleright\!\!\!\blacktriangleright_{\mu_1} j_1 \blacktriangleright\!\!\!\blacktriangleright_{\mu_2} j_2$, which means $\blacktriangleright\!\!\!\blacktriangleright$ also has contradictory rights, yielding a contradiction. $\square$

Proof of Proposition 2: Suppose on the contrary that the proper exclusion right core allocation μ were not efficient. Then there would exist another allocation ν such that $\nu(i) \succeq_i \mu(i)$ for all $i \in I$ and $\nu(i) \succ_i \mu(i)$ for some $i \in I$. Let C be the set of agents that become strictly better off at ν than at μ. Then coalition C can block μ through ν by Lemma 1. It contradicts that μ is a core allocation.

Let μ be an efficient allocation. Suppose on the contrary that $\mu(i)$ were not individually rational. Then the set $J = \{j \in I \mid s_0 \succ_j \mu(j)\}$ would be nonempty. Define a new allocation ν by $\nu(i) = s_0$ for every $i \in J$ and $\nu(i) = \mu(i)$ for every $i \in I \setminus J$. Clearly, μ is Pareto dominated by ν, contradicting that μ is efficient. $\square$

Proof of Proposition 3: Suppose on the contrary that the proper core allocation μ were not properly fair. Then there would exist two agents $i, j \in I$ such that i properly envies $\mu(j) \succ_i \mu(i)$ and $i \blacktriangleright^p_\mu j$. Then i can block μ by directly excluding j from $\mu(j)$, which contradicts that μ is in the proper core.

Suppose on the contrary that the proper core allocation μ were not weakly fair. Then there would exist two agents i and j such that $i \rhd_s j$ for all $s \in S$ but $\mu(j) \succ_i \mu(i)$. Following Lemma 1, we have $j \blacktriangleright\!\!\!\blacktriangleright_\mu j$. It is either (1) that $j \blacktriangleright_\mu j$ holds or (2) there exist a sequence of agents such that $j \blacktriangleright_\mu j_1 \blacktriangleright_\mu \cdots \blacktriangleright_\mu j_L \blacktriangleright_\mu j$. Recall that the direct exclusion right scheme $\blacktriangleright^p$ respects the priority structure. In the first case, we have $i \rhd_{\mu(j)} j \blacktriangleright_\mu j$ and thus $i \blacktriangleright_\mu j$. In the second case, we have $i \rhd_{\mu(j_1)} j \blacktriangleright_\mu j_1$ and thus $i \blacktriangleright_\mu j_1 \blacktriangleright_\mu \cdots, \blacktriangleright_\mu j_L \blacktriangleright_\mu j$. In either case, we have $i \blacktriangleright\!\!\!\blacktriangleright_\mu j$. Then i can block μ by directly or indirectly excluding j from $\mu(j)$, which contradicts the fact that μ is in the proper core. $\square$

Proof of Proposition 5: **The 'if' part:** Suppose the direct scheme $\blacktriangleright$ is represented by a threshold scheme θ. For every $\mu \in \mathcal{A}$ and every agent $i \in I$, an agent j has a right to exclude i only if $j \trianglerighteq_{\mu(i)} \theta_\mu(i) \trianglerighteq_{\mu(i)} i$, so the requirement (A1) is satisfied. Clearly, $k \trianglerighteq_{\mu(i)} j \blacktriangleright_\mu i$ implies $k \trianglerighteq_{\mu(i)} \theta_\mu(i)$ and $k \blacktriangleright_\mu i$. The requirement (A2) is also satisfied. Therefore, $\blacktriangleright$ respects the priority structure.

The 'only if' part: Suppose that the direct scheme $\blacktriangleright$ respects the priority structure $\rhd$. For every allocation $\mu \in \mathcal{A}$ and every agent $i \in I$, we define $\theta_\mu(i) = \min_{\rhd_{\mu(i)}}\{j \in I \mid j \blacktriangleright_\mu i\}$. Since $\blacktriangleright$ respects the priority structure, $\theta_\mu(i)$ who has an exclusion right to i must have a relatively higher priority $\theta_\mu(i) \trianglerighteq_{\mu(i)} i$. Furthermore, every $k \rhd_{\mu(i)} \theta_\mu(i)$ also has a right to exclude i. In summary, θ_μ is the threshold representing $\blacktriangleright$. $\square$

Proof of Theorem 2: To have a better understanding of the properties of the TPC algorithm and the related exclusion right system, we introduce a generalized TPC algorithm, in which a group of agents delay their pointing. For the generalized TPC algorithm, fix a group of agents $A \subseteq I$ and let the agents in A not point to any object. Then in each step of Phase 1, every vertex in $(I \cup S) \setminus A$ is either involved in a cycle or linked to an agent $i \in A$ through a directed path. [6] Remove all cycles and repeat this operation until

[6]Vertices a and i are said to *be linked* if there exists a sequence of alternating agents and objects $(a, \mu(a), j_1, s_1, \ldots, j_L, s_L, i)$ for a being an agent or $(a, j_1, s_1, \ldots, j_L, s_L, i)$ for a being an object such that agent a points to $\mu(a)$ and $\mu(a)$ points to j_1, or object a points to j_1, and then for both cases agent j_1 points to

all remaining agents are linked to some agent in A. Finally, in Phase 2, we implement the TPC algorithm for the remaining agents and objects.

The Generalized Top Priority Cycle Algorithm Given $A \subseteq I$

- Phase 1: For any given allocation μ, define $I^0 = \{i \in I \mid \mu(i) = s_0\}$ and $S^0 = \{s \in S \mid \mu^{-1}(s) = i_0\}$. For each $i \in I^0$, set $\theta^g_\mu(i) = \emptyset$. Remove $I^0 \setminus A$ and S^0. Then set $t = 1$, $I^1 = I \setminus I^0$, and $S^1 = S \setminus S^0$.
  - At each step $t \geq 1$, every remaining agent $i \in I^t \setminus A$ points to $\mu(i)$. Every remaining object $s \in S^t$ points to the remaining agent who has the highest priority on s among agents in I^t. If there exists any cycle, let X^t be the set of agents and objects which are in some cycle. For every agent $i \in X^t$, set $\theta^g_\mu(i)$ to be the agent to which $\mu(i)$ points. Remove all the cycles and set $I^{t+1} = I^t \setminus X^t$ and $S^{t+1} = S^t \setminus X^t$. Let $t = t + 1$ and repeat the operation until there is no cycle. If there is no cycle, go to Phase 2. Let r be the last step of Phase 1.
- Phase 2: Remove the set $I^0 \cap A$. Set $t = r + 1$, $I^t = I^r \setminus I^0$, and $S^t = S^r$.
  - At each step $t \geq r + 1$, every remaining agent $i \in I^t$ points to $\mu(i)$. Every remaining object $s \in S^t$ points to the remaining agent who has the highest priority on s among agents in I^t. If there exists any cycle, let X^t be the set of agents and objects which are in some cycle. For every agent $i \in X^t$, set $\theta^g_\mu(i)$ to be the agent to which $\mu(i)$ points. Remove all the cycles and set $I^{t+1} = I^t \setminus X^t$ and $S^{t+1} = S^t \setminus X^t$. Let $t = t + 1$ and repeat the operation until all agents and objects are removed.

.

Note that the generalized TPC algorithm reduces to the TPC algorithm when $A = \emptyset$. The following lemma illustrates the relation between the TPC algorithm and the generalized TPC algorithm.

object s_1, ..., agent j_L points to object s_L, and s_L points to agent i. This sequence is called a *directed path* from a to i.

Lemma 4 *For any group of agents $A \subseteq I$ and any allocation μ, the outcome generated by the generalized TPC algorithm is the same as the outcome generated by the TPC algorithm.*

Proof. To distinguish the two algorithms, we add wave symbols to things related to the generalized TPC algorithm.

At each step $t \leq r$ of the generalized TPC algorithm, every vertex in $(I \setminus \tilde{I}^r) \cup (S \setminus \tilde{S}^r)$ must point to the same vertex as it does at step t of the TPC algorithm. So in the generalized TPC algorithm, the vertices in $(I \setminus \tilde{I}^r) \cup (S \setminus \tilde{S}^r)$ must form the same cycles among themselves as they do in the TPC algorithm. That is $\tilde{\theta}^g_\mu(i) = \theta^g_\mu(i)$ for every $i \in I \setminus \tilde{I}^r$. This also implies that all vertices in $\tilde{I}^{r+1} \cup \tilde{S}^{r+1}$ form cycles among themselves in the two algorithms. We now prove that they have the same cycles by induction.

Consider a general step $t \geq r+1$ of the generalized TPC algorithm. We prove that every agent in $I \setminus \tilde{I}^{t-1}$ leaves from the same cycle in both algorithms implies that every agent $i \in \tilde{I}^t \setminus \tilde{I}^{t+1}$ is involved in the same cycle when she is removed from both algorithms. Let $\{i = i_1 = i_L, \mu(i_1), i_2, \mu(i_2) \ldots, \mu(i_{L-1}), i_1\}$ be the cycle involving i at step t of the generalized TPC algorithm. Then for every $\ell \in \{2, \ldots, L-1\}$, i_ℓ has the highest priority on $\mu(i_{\ell-1})$ among agents in $\tilde{I}^t$. Consider the step at which i leaves from the TPC algorithm. We know that i_1 cannot leave when $\mu(i_1)$ points to an agent in $I \setminus \tilde{I}^{t-1}$, and i_2 has the highest priority on $\mu(i_1)$ among agents in $\tilde{I}^{t+1}$, so i_1 and $\mu(i_1)$ should remain as long as i_2 and $\mu(i_2)$ remain. Inductively, i_2 and $\mu(i_2)$ should remain when i_3 and $\mu(i_3)$ remain, and so on. When i_1 leaves from the TPC algorithm, all elements of the cycle remain and form the same cycle as in the generalized TPC algorithm. Hence, we have proved that every agent $j \in \tilde{I}^t \setminus \tilde{I}^{t+1}$ leaves from the same cycle in both algorithms and $\theta^g_\mu(j) = \tilde{\theta}^g_\mu(j)$. □

The TPC algorithm has the following properties.

Lemma 5 *Assume that the TPC algorithm is implemented for allocation μ. If object $\mu(j)$ has pointed to agent i at some step, then $i \trianglerighteq_{\mu(j)} \theta^g_\mu(j)$ and $i \blacktriangleright^g_\mu j$. Furthermore, if agent j is linked to agent i at some step, then $i \blacktriangleright\!\!\!\blacktriangleright^g_\mu j$.*

Proof. Suppose that $\mu(j)$ points to i at some step t. Then i has the highest priority on $\mu(j)$ among agents in I^t. When $\mu(j)$ leaves at step $\tau \geq t$, it should point to an agent $k \in I^\tau \subseteq I^t$. So, we have $i \trianglerighteq_{\mu(j)} k = \theta^g_\mu(j)$ and thus $i \blacktriangleright^g_\mu j$.

Suppose that j is linked to i at some step t. Let $(j, \mu(j), j_1, \mu(j_1), \ldots, j_L, \mu(j_L), i)$ be the path. By the former part of the lemma, we have $i \blacktriangleright^g_\mu j_L \blacktriangleright^g_\mu \cdots \blacktriangleright^g_\mu j_1 \blacktriangleright^g_\mu j$, that is $i \blacktriangleright\!\!\!\blacktriangleright^g_\mu j$.

$\square$

Lemma 6 *Assume that the TPC algorithm is implemented for allocation μ. If agent i leaves later than agent j, then i cannot have a right to exclude j at μ. If agents i and j leave at the same step but they are in different cycles, then they cannot have a right to exclude each other.*

Proof. We prove the lemma by induction. Suppose that the lemma is true for all agents who leave before $t \geq 1$. Note that the basic case is valid since no one leaves before step $t = 1$. Suppose that agent j leaves at step t. Let $X = \{j = j_1, \mu(j_1), \ldots, j_L, \mu(j_L), j\}$ be the cycle involving j when j leaves. For any agent $i' \in I^t \setminus X$ and $j_\ell \in X$, we have $\theta^g_\mu(j_\ell) \rhd_{\mu(j_\ell)} i'$, and thus $i' \not\blacktriangleright^g_\mu j_\ell$. Now consider an agent i who leaves later or at step t but is not in the cycle. If there is a sequence of agents such that $i \blacktriangleright^g_\mu i_1 \blacktriangleright^g_\mu \ldots \blacktriangleright^g_\mu i_L \blacktriangleright^g_\mu j$. Then $i_L \blacktriangleright^g_\mu j$ implies that i_L leaves before step t or in the cycle X, i.e., $i_L \in (I \setminus I^t) \cup X$. Similarly, $i_{L-1} \blacktriangleright^g_\mu i_L$ implies $i_{L-1} \in (I \setminus I^t) \cup X$. Repeat this argument. Finally, we get $i \in (I \setminus I^t) \cup X$, contradicting the assumption on agent i. Consequently, we have proved that the argument is also true for agent j leaving at step t. $\square$

Now we are ready to prove Theorem 2.

Proof. **Part I:** The system $(\blacktriangleright^g, \blacktriangleright\!\!\!\blacktriangleright^g)$ satisfies the two properties.

Proof of (P1). $\blacktriangleright^g$ is characterized by a threshold scheme θ^g. By Proposition 5, $\blacktriangleright^g$ respects priorities.

Proof of (P2). We first show that $\blacktriangleright\!\!\!\blacktriangleright^g$ does not contain any contradictory rights. Suppose on the contrary that there existed contradictory rights. Then there would exist two different agents i, j and two allocations μ, ν such that $s = \mu(i) = \nu(j)$, $\mu(k) = \nu(k)$ for all other agents $k \in I \setminus \{i, j\}$ and $j \blacktriangleright\!\!\!\blacktriangleright^g_\mu i \blacktriangleright\!\!\!\blacktriangleright^g_\nu j$.

Set $A = \{i, j\}$ and implement the generalized TPC algorithm for the allocations μ and ν. Since $\mu(k) = \nu(k)$ for all $k \in I \setminus A$, there is no difference between the two allocations in Phase 1. At the last step r of Phase 1, each remaining agent in I^r should be linked to either i or j. Let I^r_i be the set of agents that are linked to agent i, and let I^r_j be the set of agents that are linked to agent j. Then we have $I^r_i \cup I^r_i = I^r$ and $I^r_i \cap I^r_j = \emptyset$. Let k be the agent who has the highest priority on s among agents in I^r. If $k \in I_i^r$, then at step $r+1$ of the algorithm for allocation μ, i points to r and r points to k, and i is in a cycle without j. By Lemma 6, we have $j \not\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g} i$. This is a contradiction. Similarly, if $k \in I_j^r$, then at step $r+1$ of the algorithm for allocation ν, j points to r and r points to k, and j is in a cycle without i. It is another contradiction $i \not\blacktriangleright\!\!\!\blacktriangleright_{\nu}^{g} j$.

We now show that any larger derived exclusion right scheme contains contradictory rights. Let $\blacktriangleright\!\!\!\blacktriangleright'$ be a derived scheme that is strictly larger than $\blacktriangleright\!\!\!\blacktriangleright^{g}$. Then there exist at least one allocation μ and two agents i and j such that $i \blacktriangleright\!\!\!\blacktriangleright'_{\mu} j$ but $i \not\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g} j$. Let ν be the allocation such that $\nu(i) = \mu(j) = s$, $\nu(j) = s_0$ and $\nu(k) = \mu(k)$ for all $k \in I \setminus \{i, j\}$. Implement the generalized TPC algorithm for the two allocations μ and ν with $A = \{i, j\}$. Since $\mu(k) = \nu(k)$ for all $k \in I \setminus A$, there is no difference between the two algorithms in Phase 1. At the last step r of Phase 1, each remaining object in S^r should be linked to i or j. Specifically, s should be linked to j rather than i. Otherwise, in the algorithm for μ, the cycle involving j and s should include i and thus $i \blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g} j$, yielding a contradiction. Now consider step r of the algorithm for ν. Agent i points to s, and s is also linked to j through a directed path, says $(s, j_1, \nu(j_1), \ldots, j_L, \nu(j_L), j)$. By Lemma 5, we have $j \blacktriangleright_{\nu}^{g} j_L \blacktriangleright_{\mu}^{g} \cdots \blacktriangleright_{\nu}^{g} j_1 \blacktriangleright_{\nu}^{g} i$. That is, $j \blacktriangleright\!\!\!\blacktriangleright_{\nu}^{g} i$. Recall that $\blacktriangleright\!\!\!\blacktriangleright'$ is larger than $\blacktriangleright\!\!\!\blacktriangleright^{g}$, so we have $j \blacktriangleright\!\!\!\blacktriangleright'_{\nu} i$. Now we get contradictory rights $j \blacktriangleright\!\!\!\blacktriangleright'_{\nu} i \blacktriangleright\!\!\!\blacktriangleright'_{\mu} j$.

Part II: An exclusion right system $(\blacktriangleright^{p}, \blacktriangleright\!\!\!\blacktriangleright^{p})$ satisfies the two properties only if $\blacktriangleright^{p}$ respects the priority structure and $\blacktriangleright\!\!\!\blacktriangleright^{p} = \blacktriangleright\!\!\!\blacktriangleright^{g}$.

Suppose $(\blacktriangleright^{p}, \blacktriangleright\!\!\!\blacktriangleright^{p})$ satisfies the two properties. There is a threshold scheme θ^p that represents the scheme $\blacktriangleright^{p}$. Let us prove $\blacktriangleright\!\!\!\blacktriangleright^{p} = \blacktriangleright\!\!\!\blacktriangleright^{g}$ by induction. Define $\mathcal{A}^m$ as the set of allocations, each of which assigns no more than m real objects to agents. Formally,

$$\mathcal{A}^m = \{\mu \in \mathcal{A} \mid |\{i \in I \mid \mu(i) \neq s_0\}| \leq m\}.$$

Basic case: For the unique allocation $\mu_0 \in \mathcal{A}^0$, no agent can be excluded by another agent. Clearly, we have $\blacktriangleright\!\!\!\blacktriangleright_{\mu_0}^{p} = \blacktriangleright\!\!\!\blacktriangleright_{\mu_0}^{g} = \emptyset$. For each $\mu \in \mathcal{A}^1 \setminus \mathcal{A}^0$, let i be the unique assigned agent. Setting $\theta_\mu^p(i) = i$ and $\theta_\mu^p(j) = \emptyset$ for every $j \neq i$ would not create any contradictory rights. Therefore, the two schemes $\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{p}$ and $\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g}$ should be the same at allocation $\mu \in \mathcal{A}^1$.

Induction steps. Given that $\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{p} = \blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g}$ for all $\mu \in \mathcal{A}^m$, let us prove that $\blacktriangleright\!\!\!\blacktriangleright_{\mu}^{p} = \blacktriangleright\!\!\!\blacktriangleright_{\mu}^{g}$ for all $\mu \in \mathcal{A}^{m+1}$.

Suppose that $\theta^p_\mu(i) \rhd_{\mu(i)} \theta^g_\mu(i)$ for some $\mu \in \mathcal{A}^{m+1}$ and some assigned agent $i \in I$. By Lemma 1, $i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu i$ is a necessary condition for $\mathrel{\blacktriangleright\!\!\triangleright}^p$ to satisfy the MAXISC exclusion right property. Agent i does not have a direct exclusion right to herself under $\mathrel{\blacktriangleright\!\!\triangleright}^p$ since $\theta^p_\mu(i) \rhd_{\mu(i)} \theta^g_\mu(i) \trianglerighteq_{\mu(i)} i$. Therefore, there should be an agent j such that $i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu j$ and $j \trianglerighteq_{\mu(i)} \theta^p_\mu(i)$. Consider the allocation ν defined by $\nu(i) = \mu(j)$, $\nu(j) = s_0$, and $\nu(k) = \mu(k)$ for all $k \in I \setminus \{i, j\}$. It is clear that $\nu \in \mathcal{A}^m$ and therefore $\mathrel{\blacktriangleright\!\!\triangleright}^p_\nu = \mathrel{\blacktriangleright\!\!\triangleright}^g_\nu$.

Consider the TPC algorithm implemented for μ. Let t be the step at which i and $\mu(i)$ leave. At this step, $\mu(i)$ points to $\theta^g_\mu(i)$, and all agents with a higher priority on $\mu(i)$ than $\theta^g_\mu(i)$ (including agent j) have left. By Lemma 6, agent i cannot exclude j who leaves earlier than i under the scheme $\mathrel{\blacktriangleright\!\!\triangleright}^g$. Set $A = \{i, j\}$ and implement the generalized TPC algorithm for the allocations μ and ν. Since $\mu(k) = \nu(k)$ for all $k \in I \setminus A$, there is no difference between the two allocations in Phase 1 (except $\mu(i)$ leaves at step 0 of the algorithm implemented for ν). At the last step r of Phase 1, each remaining object should be linked to i or j. If the object $\mu(j) = \nu(i)$ is linked to i, then the algorithm implemented for μ implies that $i \mathrel{\blacktriangleright\!\!\triangleright}^g_\mu j$ by Lemma 5, which contradicts the above analysis. Therefore, $\mu(j) = \nu(i)$ must be linked to j, and the algorithm implemented for ν implies $j \mathrel{\blacktriangleright\!\!\triangleright}^g_\nu i$. Recall that $\mathrel{\blacktriangleright\!\!\triangleright}^g_\nu = \mathrel{\blacktriangleright\!\!\triangleright}^p_\nu$. Now $\mathrel{\blacktriangleright\!\!\triangleright}^p$ has contradictory rights as $j \mathrel{\blacktriangleright\!\!\triangleright}^p_\nu i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu j$. This is a contradiction.

We now have $\theta^g_\mu(i) \trianglerighteq_{\mu(i)} \theta^p_\mu(i)$ for all $\mu \in \mathcal{A}^{m+1}$ and all assigned agent $i \in I$. We will show that $i \mathrel{\blacktriangleright\!\!\triangleright}^g_\mu j$ implies $i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu j$. If $i \mathrel{\blacktriangleright\!\!\triangleright}^g_\mu j$, then there is a sequence of agents such that $i = i_1 \blacktriangleright^g_\mu i_2 \blacktriangleright^g_\mu \cdots \blacktriangleright^g_\mu i_L = j$. For each $\ell \in \{1, \ldots, L-1\}$, $i_\ell \blacktriangleright^g_\mu i_{\ell+1}$ implies $i_\ell \trianglerighteq_{\mu(i_{\ell+1})} \theta^g_\mu(i_{\ell+1}) \trianglerighteq_{\mu(i_{\ell+1})} \theta^p_\mu(i_{\ell+1})$, and so $i_\ell \blacktriangleright^p_\mu i_{\ell+1}$. Therefore, the same sequence implies $i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu j$.

Suppose $\mathrel{\blacktriangleright\!\!\triangleright}^p_\mu \neq \mathrel{\blacktriangleright\!\!\triangleright}^g_\mu$ for some $\mu \in \mathcal{A}^{m+1}$. Then there is at least one pair of agents $i, j \in I$ such that $i \mathrel{\blacktriangleright\!\!\triangleright}^p_\mu j$ but $i \not\mathrel{\blacktriangleright\!\!\triangleright}^g_\mu j$. Define the allocation ν by $\nu(i) = \mu(j)$, $\nu(j) = s_0$, and $\nu(k) = \mu(k)$ for all $k \in I \setminus \{i, j\}$. Note that $\nu \in \mathcal{A}^m$ if $\mu(i) \neq s_0$, and that $\nu \in \mathcal{A}^{m+1}$ if $\mu(i) = s_0$. Set $A = \{i, j\}$ and run the generalized TPC algorithms for the allocations μ and ν. Since $\mu(k) = \nu(k)$ for all $k \in I \setminus A$, there is no difference between the two allocations in Phase 1 (except $\mu(i)$ leaves at step 0 of the algorithm implemented for ν if $\mu(i) \neq s_0$). At the last step of Phase 1, each remaining object should be linked to i or j. If the object $\mu(j) = \nu(i)$ is linked to i, then the algorithm implemented for μ implies $i \mathrel{\blacktriangleright\!\!\triangleright}^g_\mu j$

by Lemma 5, which contradicts the assumption $i \not\blacktriangleright\!\!\triangleright^g_\mu j$. Therefore, $\mu(j) = \nu(i)$ must be linked to j, and the algorithm implemented for ν implies $j \blacktriangleright\!\!\triangleright^g_\nu i$. If $\nu \in \mathcal{A}^m$, we have $j \blacktriangleright\!\!\triangleright^p_\nu i$ since $\blacktriangleright\!\!\triangleright^p_\nu = \blacktriangleright\!\!\triangleright^g_\nu$. Similarly, if $\nu \in \mathcal{A}^{m+1}$, we also have $j \blacktriangleright\!\!\triangleright^p_\nu i$. In either case, we have $j \blacktriangleright\!\!\triangleright^p_\nu i$. Now $\blacktriangleright\!\!\triangleright^p$ contains contradictory rights as $j \blacktriangleright\!\!\triangleright^p_\nu i \blacktriangleright\!\!\triangleright^p_\mu j$. This contradicts the fact that $\blacktriangleright\!\!\triangleright$ is self-consistent.

As a result, we have proved that $\blacktriangleright\!\!\triangleright_\mu = \blacktriangleright\!\!\triangleright^g_\mu$ for all $\mu \in \mathcal{A}^{m+1}$. □

Proof of Proposition 6: Implement the generalized TPC algorithm for the allocations μ and ν with $A = \{i, j\}$. Since $\mu(k) = \nu(k)$ for all $k \in I \setminus A$, there is no difference between the two allocations in Phase 1. At the last step r of Phase 1, each remaining object in S^r should be linked to either i or j. If $\mu(i) = \nu(j)$ is linked to i, then the algorithm implemented for ν implies $i \blacktriangleright\!\!\triangleright^g_\nu j$. By Theorem 2, we have $i \blacktriangleright\!\!\triangleright^p_\nu j$. Otherwise, $\mu(i) = \nu(j)$ is linked to j, then the algorithm implemented for μ implies $j \blacktriangleright\!\!\triangleright^g_\mu i$ and thus $j \blacktriangleright\!\!\triangleright^p_\mu i$. □

Proof of Lemma 2: We prove it by induction. For a general step $t \geq 1$, suppose that the lemma is true for every step $\tau < t$ and every agent $i \in X^\tau$. Note that the basic case is valid since no one leaves before step $t = 1$. We now show that the lemma is also true for every agent $i \in X^t$. Here we see that agents and objects in $I^t \cup S^t$ should form cycles among themselves in the two algorithms. If i points to s_0 and leaves without a cycle at step t of the TTC algorithm, then $\mu^*(i) = s_0$ implies that i leaves at the preparing stage of the TPC algorithm without a cycle. Otherwise, i is involved in a cycle and let $\{i = i_1, \mu^*(i_1), i_2, \mu^*(i_2) \ldots, i_L = i_1\}$ be the cycle. For every $\ell \in \{2, \ldots, L\}$, i_ℓ has the highest priority on $\mu^*(i_{\ell-1})$ among agents in I^t. In the TPC algorithm, $\mu^*(i_{\ell-1})$ cannot leave pointing to an agent in $I \setminus I^t$, and i_ℓ has the highest priority on $\mu^*(i_{\ell-1})$ among agents in I^t, so that i_ℓ remains implies that $\mu^*(i_{\ell-1})$ and $i_{\ell-1}$ remain. Thus, when i leaves from the TPC algorithm, all elements of the cycle remain, and should form the same cycle as in the TTC algorithm.

By the induction, we find that the two algorithms share the same set of cycles. In addition, the threshold of each assigned agent $i \in I$ is the agent pointed by $\mu^*(i)$ when i leaves the TTC algorithm. □

Proof of Theorem 3: Fix an economy $\langle I, S, \succ, \rhd \rangle$. Let $(\blacktriangleright^p, \blacktriangleright\!\!\triangleright^p)$ be the proper exclusion right system generated by the TPC algorithm, and let μ^* be the allocation produced by the TTC algorithm.

Part I: μ^* is in the proper exclusion right core.

Suppose on the contrary that μ^* were not in the proper core. Then there would exist a coalition $C \subsetneq I$ that blocks μ^* through some allocation ν such that $\nu(i) \succ_i \mu^*(i)$ for all $i \in C$ and $\mu^*(j) \succ_j \nu(j)$ implies that there is an agent $i \in C$ such that $i \blacktriangleright^p_{\mu^*} j$.

We first show that if j leaves earlier than i in the TTC algorithm, then i cannot exclude j under $\blacktriangleright^p_{\mu^*}$. Suppose that j leaves at step t of the TTC algorithm and i remains at step $t+1$. By Lemma 2 we know that $\theta^p_{\mu^*}(j)$ also leaves at step t. If $i \rhd_{\mu^*(j)} \theta^p_{\mu^*}(j)$, we have a contradiction that $\mu^*(j)$ should point to i rather than $\theta^p_{\mu^*}(j)$. It must be $\theta^p_{\mu^*}(j) \rhd_{\mu^*(j)} i$. Similarly, we have that for each $j' \in I \setminus I^{t+1}$ and $i' \in I^{t+1}$, $i' \not\blacktriangleright^p_{\mu^*} j'$. Consequently, we have $i \not\blacktriangleright^p_{\mu^*} j$.

We show that there exists an agent who is strictly worse off at ν and leaves earlier than anyone else in the coalition in the TTC algorithm. Let $i \in C$ be the agent who was the first from the coalition to leave the TTC algorithm. Let t_0 be the step at which i leaves. If $t_0 = 1$, then i receives the most preferred object among objects in S at μ^* and cannot be strictly better off at ν. So we have $t_0 > 1$. Since i strictly prefers $\nu(i)$ to $\mu^*(i)$ and $\mu^*(i)$ is i's most preferred object among objects in S^{t_0}, $\nu(i) \notin S^{t_0}$ must leave earlier than i. Let $t_1 < t_0$ be the step at which $\nu(i)$ leaves. According to the TTC algorithm, $\nu(i)$ must be assigned. Let j_1 be the agent such that $\mu^*(j_1) = \nu(i)$. Then j_1 also leaves at step t_1. If j_1 strictly prefers $\mu^*(j_1)$ to $\nu(j_1)$, then j_1 is the agent we want to find. Otherwise, j_1 strictly prefers $\nu(j_1)$ to $\mu^*(j_1)$, and $\mu^*(j_1)$ is j_1's most preferred object among objects in S^{t_1}, so $\nu(j_1) \notin S^{t_1}$ should leave at an even earlier step $t_2 < t_1$. Let j_2 be the agent such that $\mu^*(j_2) = \nu(j_1)$ who also leaves at step $t_2 < t_1$. If j_2 strictly prefers $\mu^*(j_2)$ to $\nu(j_2)$, then j_2 is the agent we want to find. Otherwise, applying the same argument, we find $j_3 = \mu^{*-1}(\nu(j_2))$ who leaves at step $t_3 < t_2$. Repeat this argument. There are finitely many steps. When we find an agent j_L who leaves at the first step by repeating the argument, j_L must be strictly worse off at ν because $\mu^*(j_L)$ is the most preferred object among all objects in $S^1 = S$. Thus, j_L is the agent we want to find. Consequently, there exists an agent who is strictly worse off at ν and leaves earlier than any other member of the coalition, and we use j to denote the agent.

Now, j leaves earlier than every agent in the coalition, so there does not exist an agent

$i \in C$ such that $i \blacktriangleright\!\!\triangleright^p_{\mu^*} j$. This contradicts that C can block μ^* through ν.

Part II: μ^* is the unique proper exclusion core allocation.

For any allocation μ' different from μ^*, the set $J = \{j \in I \mid \mu'(j) \neq \mu^*(j)\}$ is not empty. Let $j_1 \in J$ be the agent who leaves earliest in the TTC algorithm among the agents in J. Let t be the step at which j_1 leaves, and let $X = \{j_1, \mu^*(j_1) = s_1, \ldots, j_L, \mu^*(j_L) = s_L, j_1\}$ be the cycle involving j_1 in the TTC algorithm. We now show that the coalition $C = J \cap X$ can block μ'. Note that if agent i leaves earlier than step t, i.e., $i \in I \setminus I^t$, then $i \notin J$ and $\mu'(i) = \mu^*(i)$.

To obtain the exclusion right system at μ', we implement the generalized TPC algorithm for μ' with $A = C$. It is clear that every agent or object that leaves before step t of the TTC algorithm (i.e., $a \in (I \setminus I^t) \cup (S \setminus S^t)$) leaves from the same cycle in Phase 1 of the generalized TPC algorithm. We omit the formal proof of the statement since it is similar to the proof of Lemma 2.

Now consider the step $r+1$ of Phase 2 of the generalized TPC algorithm. Agent $j_1 \in C$ remains at step $r+1$ of the generalized TPC algorithm and has the highest priority on $s_L = \mu^*(j_L)$ among agents in I^t. Agent s_L would not leave by pointing to an agent in $I \setminus I^t$, so s_L also remains. Furthermore, agents in $I \setminus I^t$ have left in Phase 1, so s_L should point to j_1 at step $s+1$ of Phase 2. That is, $j_1 \blacktriangleright^p_{\mu'} \mu'^{-1}(s_L)$. If $\mu'(j_L) = \mu^*(j_L)$, then $j_L = \mu'^{-1}(s_L)$ remains. For a similar reason, $\mu^*(j_{L-1})$ remains and points to j_L at step $r+1$. That is, $j_L \blacktriangleright^p_{\mu'} j_{L-1}$ and thus $j_1 \blacktriangleright\!\!\triangleright^p_{\mu'} j_{L-1}$. If $\mu'(j_L) \neq \mu^*(j_L)$, then $j_L \in J \cap X = C$ also remains. We also have that s_{L-1} remains and points to j_L at step $r+1$. Then we have $j_L \in C$ and $j_L \blacktriangleright\!\!\triangleright^p_{\mu'} \mu'^{-1}(s_{L-1})$. Inductively, we can prove that, for every $\ell \in \{1, \ldots, L\}$, there exists an agent $i \in C$ such that $i \blacktriangleright\!\!\triangleright^p_{\mu'} \mu'^{-1}(s_\ell)$. Now consider the allocation ν defined by

$$\nu(i) = \begin{cases} \mu^*(i), & \text{if } i \in X; \\ s_0, & \text{if } i \notin X \text{ but } \mu'(i) \in X; \\ \mu'(i), & \text{otherwise.} \end{cases}$$

For each $i \in C = J \cap X$, $\nu(i)$ is her most preferred object among objects in S^t, and she receives a less preferred object in S^t at μ', so she strictly prefers $\nu(i) = \mu^*(i)$ to $\mu'(i)$. For each j such that $j \notin X$ but $\mu'(j) \in X$, we can find an agent $i \in C$ such that $i \blacktriangleright\!\!\triangleright^p_{\mu'} j$. Therefore, ν is valid for coalition C to block μ'. We are done. $\square$

Proof of Theorem 1: The first half follows from Theorem 2 and the second half follows from Theorem 3. □

Proof of Proposition 4: By Theorem 3 the allocation generated by the TTC algorithm, denoted by μ^*, is the unique proper exclusion right core allocation. Construct a price vector p^* as follows: For each object $s \in S$ which leaves at step $t \geq 1$ of the TTC algorithm, set $p^*(s) = (1/2)^t$. For each agent $i \in I$ who leaves at step $t \geq 1$ of the TTC algorithm, set $y^*(i) = (1/2)^t$. For each unassigned object s, set $p^*(s) = 0$. We prove that (y^*, p^*, μ^*) is a competitive equilibrium. First, we have $p^*(s) = 0$ for each unassigned object $s \in S$ by the definition of p^*. Second, for each pair $i \blacktriangleright_\mu j$, we know that j cannot leave earlier than i by Lemmas 2 and 6, so $y^*(j) \leq y^*(i)$ is consistent with the proper direct exclusion right scheme $\blacktriangleright_\mu$. Finally, for each agent $i \in I$ who leaves at step $t \geq 1$ of the TTC algorithm, she cannot afford an object that leaves earlier than step t. The remaining objects are all affordable, and $\mu^*(i)$ is the most preferred object among the remaining ones. So $\mu^*(i) \in D^i(p^*, y^*)$. □

Proof of Theorem 4: We first check the "if" part. Theorem 3 shows that the TTC algorithm generates the unique proper exclusion right core allocation, which is properly fair and Pareto efficient by Propositions 2 and 3. Then by Proposition 7, the TTC mechanism is strategy-proof.

Let's prove the "only if" part. Suppose that mechanism ϕ is properly fair, Pareto efficient, and strategy-proof. For any given economy $\langle I, S, \succ, \rhd \rangle$, let $\mu^* = TTC_{pec}(\succ)$. We will prove that $\phi(\succ)(i) = \mu^*(i)$ for all $i \in I$. We use the notation $(I^t, S^t, X^t)_{1 \leq t \leq T}$ in the TTC algorithm. We prove the result by induction.

For a general step $t \geq 1$, given that $\phi(\succ_{I \setminus I^t}, \succ''_{I^t})(i) = \mu^*(i)$ is true for any $i \in I \setminus I^t$ and any $\succ''_{I^t}$, we prove that $\phi(\succ_{I \setminus I^{t+1}}, \succ''_{I^{t+1}})(i) = \mu^*(i)$ is also true for any agent $i \in X^t$ and any $\succ''_{I^{t+1}}$. If $\mu^*(i) = s_0$, then s_0 is i's most preferred object among objects in $S^t \cup \{s_0\}$. Agent i cannot get an object from $S \setminus S^t$, which are assigned to agents in $I \setminus I^t$, so i should receive s_0 by Pareto efficiency. Consider the other case where i leaves from the cycle $Y = \{i = i_1, \mu^*(i_1), i_2, \mu^*(i_2) \ldots, i_L = i_1\}$.

Define $\mathcal{A}' = \{\mu \in \mathcal{A} \mid \mu(i) = \mu^*(i) \text{ for all } i \in I \setminus I^t\}$. By the inductive condition, we have $\phi(\succ_{I \setminus I^{t+1}}, \succ''_{I^{t+1}}) \in \mathcal{A}'$. When we consider the proper exclusion right system for any allocation $\mu \in \mathcal{A}'$, it is easy to see that the cycles appearing at step $\tau < t$ of the TPC algorithm implemented for μ^* will also appear at step τ of the TPC algorithm implemented for μ. By Lemma 6, agents in I^t do not have any exclusion right to those agents who have left before step t. Similarly, the agent who have left before step t is not the threshold of any remaining agent $i \in I^t$. Thus, $\theta^p_\mu(i) \in I^t$ for any $i \in I^t$ and any $\mu \in \mathcal{A}'$. For any agent $i_\ell \in Y$, i_ℓ has the highest priority on $\mu^*(i_{\ell-1})$ among agents in I^t, and $\theta^p_\mu(\mu^*(i_{\ell-1})) \in I^t$ for any $\mu \in \mathcal{A}'$, so we have $i_\ell \trianglerighteq_{\mu^*(i_{\ell-1})} \theta^p_\mu(\mu^*(i_{\ell-1}))$ and thus i_ℓ has the right to directly exclude the occupant of $\mu^*(i_{\ell-1})$. What's more, i_ℓ likes $\mu^*(i_\ell)$ most among objects in $S^t \cup \{s_0\}$. For every agent $i_\ell \in Y$, consider the preference relation

$$\succ'_{i_\ell}: \underbrace{\mu^*(i_\ell), \mu^*(i_{\ell-1})}_{\text{truncation of } \succ_{i_\ell}}, \ldots, \tag{A.2}$$

which removes the objects before $\mu^*(i_\ell)$ and between $\mu^*(i_\ell)$ and $\mu^*(i_{\ell-1})$.

We show the statement $\phi(\succ'_Z, \succ_{I\setminus(I^{t+1}\cup Z)}, \succ''_{I^{t+1}})(i_\ell) = \mu^*(i_\ell)$ is true for any subset $Z \subseteq Y$, any $i_\ell \in Y$, and any $\succ''_{I^{t+1}}$.

First, consider the case $Z = Y$. Let $\mu^Z \equiv \phi(\succ'_Z, \succ_{I\setminus(I^{t+1}\cup Z)}, \succ''_{I^{t+1}})$. Observe that $\mu^Z \in \mathcal{A}'$ and thus $\mu^*(i_{\ell-1}) \in \omega^p_{\mu^Z}(i_\ell)$ for all $i_\ell \in Y$. Since ϕ satisfies Pareto efficiency, we have $\mu^Z(i_\ell) \succeq'_{i_\ell} \mu^*(i_{\ell-1})$. If $\mu^Z(i_\ell) = \mu^*(i_{\ell-1})$ for some $i_\ell \in Y$, then $\mu^Z(i_\ell) = \mu^*(i_{\ell-1})$ for all $i_\ell \in Y$, and μ^Z is Pareto dominated by assigning $\mu^*(i_\ell)$ to i_ℓ for all $i_\ell \in Y$ without changing other agents' assignments. So it holds that $\mu^Z(i_\ell) = \mu^*(i_\ell)$.

Then, given that the statement is true for any $Z \subseteq Y$ such that $m < |Z| \leq |Y|$, we prove that the statement is also true for $Z \subseteq Y$ such that $|Z| = m$. Let $\mu^Z \equiv \phi(\succ'_Z, \succ_{I\setminus(I^{t+1}\cup Z)}, \succ''_{I^{t+1}})$. Suppose $\mu^Z(i_\ell) \neq \mu^*(i_\ell)$ for some $i_\ell \in Y$.

If $i_\ell \notin Z$, then i_ℓ can obtain her most preferred object $\mu^*(i_\ell)$ by misreporting $\succ'_{i_\ell}$, which contradicts that ϕ is strategy-proof. If $i_\ell \in Z$, then i_ℓ reports the truncated preference $\succ'_{i_\ell}$ and receives her second choice $\mu^Z(i_\ell) = \mu^*(i_{\ell-1})$ for the requirement of no proper envy. Thus, the next agent $i_{\ell-1}$ cannot receive her most preferred object $\mu^Z(i_{\ell-1}) \neq \mu^*(i_{\ell-1})$. Apply the same argument: If $i_{\ell-1} \notin Z$, then $i_{\ell-1}$ has an incentive to misreport $\succ'_{i_{\ell-1}}$, which contradicts that ϕ is strategy-proof; if $i_{\ell-1} \in Z$, then $i_{\ell-1}$ receives her second preferred object $\mu^Z(i_{\ell-1}) = \mu^*(i_{\ell-2})$ and the next agent $i_{\ell-2}$ cannot receive her most preferred object $\mu^Z(i_{\ell-2}) \neq \mu^*(i_{\ell-2})$. So, some agent $i_{\ell-\kappa} \in Y \setminus Z$ must have an incentive to misreport, yielding a contradiction.

Inductively, we have shown that the statement is true for any $Z \subseteq Y$, which includes the case $Z = \emptyset$. This is $\phi(\succ_{I\setminus I^{t+1}}, \succ''_{I^{t+1}})(i_\ell) = \mu^*(i_\ell)$ for all $i_\ell \in Y$. Applying the conclusion to all the cycles that leave at step t, we have $\phi(\succ_{I\setminus I^{t+1}}, \succ''_{I^{t+1}})(i) = \mu^*(i)$ for all $i \in X^t$ and all $\succ''_{I^{t+1}}$. $\square$

Proof of Lemma 3: We compare the TPC algorithm implemented for the two allocations $\tilde{\mu}_1$ and $\tilde{\mu}_2$. Since $f(\tilde{\mu}_1) = f(\tilde{\mu}_2)$, the set of students who are assigned with s_0 in $\tilde{\mu}_1$ denoted as $I^0_{\tilde{\mu}_1}$ is the same as that in $\tilde{\mu}_2$ denoted as $I^0_{\tilde{\mu}_2}$. Thus we have $I^1_{\tilde{\mu}_1} = I \setminus I^0_{\tilde{\mu}_1} = I \setminus I^0_{\tilde{\mu}_2} = I^1_{\tilde{\mu}_2}$. According to the TPC algorithm, $\theta^p_{\tilde{\mu}_1}(i) = \theta^p_{\tilde{\mu}_2}(i) = \emptyset$ for every $i \in I^0_{\tilde{\mu}_1}$. We prove the remaining part by induction.

For any step $t \geq 1$, given $I^t_{\tilde{\mu}_1} = I^t_{\tilde{\mu}_2}$, we show that $I^{t+1}_{\tilde{\mu}_1} = I^{t+1}_{\tilde{\mu}_2}$ and $\theta^p_{\tilde{\mu}_1}(i) = \theta^p_{\tilde{\mu}_2}(i)$ for every agent $i \in I^t_{\tilde{\mu}_1} \setminus I^{t+1}_{\tilde{\mu}_1}$. For every remaining agent $i \in I^t_{\tilde{\mu}_1}$ at step t of the TPC algorithm implemented for $\tilde{\mu}_1$, i points to the seat $\tilde{\mu}_1(i)$ and the seat points to the agent $j(i)$ who has the highest priority on $\tilde{\mu}_1(i)$ among agents in $I^t_{\tilde{\mu}_1}$. The same agent i remains at step t of the TPC algorithm implemented for $\tilde{\mu}_2$, and points to the seat $\tilde{\mu}_2(i)$. Since $f(\tilde{\mu}_1) = f(\tilde{\mu}_2)$, $\tilde{\mu}_2(i)$ has the same priority as $\tilde{\mu}_1(i)$ and points to the same agent $j(i)$. Therefore, if agent i is involved in a cycle at step t of the procedure for $\tilde{\mu}_1$ (i.e., $i \in I^t_{\tilde{\mu}_1} \setminus I^{t+1}_{\tilde{\mu}_1}$), then she is also in a cycle at step t of the algorithm for $\tilde{\mu}_2$ (i.e., $i \in I^t_{\tilde{\mu}_2} \setminus I^{t+1}_{\tilde{\mu}_2}$), and her threshold is the same agent (i.e., $\theta^p_{\tilde{\mu}_1}(i) = \theta^p_{\tilde{\mu}_2}(i) = j(i)$). The only difference between the formed cycles under $\tilde{\mu}_1$ and those under $\tilde{\mu}_2$ is that students in the cycles may point to different seats from the same original school so that the statement holds true. The sets of agents who leave at step t are the same for the two allocations, so the remaining agents at next step $t+1$ are the same. That is, $I^{t+1}_{\tilde{\mu}_1} = I^{t+1}_{\tilde{\mu}_2}$. $\square$

Proof of Proposition 8: *Part I.* Given the unique proper exclusion right system ($\tilde{\blacktriangleright}^p$, $\tilde{\blacktriangleright\!\!\blacktriangleright}^p$) for the alternative model, by Lemma 3, the corresponding exclusion right system ($\blacktriangleright^p$, $\blacktriangleright\!\!\blacktriangleright^p$) of the original model is unique. That is, any exclusion right system ($\blacktriangleright'$, $\blacktriangleright\!\!\blacktriangleright'$) such that $\blacktriangleright\!\!\blacktriangleright' \neq \blacktriangleright\!\!\blacktriangleright^g$ is not derived from the system ($\tilde{\blacktriangleright}^p$, $\tilde{\blacktriangleright\!\!\blacktriangleright}^p$). We have the following claims:

Claim (1). If ($\tilde{\blacktriangleright}'$, $\tilde{\blacktriangleright\!\!\blacktriangleright}'$) respects priorities, clearly the derived system ($\blacktriangleright$, $\blacktriangleright\!\!\blacktriangleright$) respects priorities.

Claim (2). If ($\tilde{\blacktriangleright}'$, $\tilde{\blacktriangleright\!\!\blacktriangleright}'$) has MAXISC exclusion rights, the derived system ($\blacktriangleright'$, $\blacktriangleright\!\!\blacktriangleright'$) has MAXISC exclusion rights. Suppose the derived system ($\blacktriangleright'$, $\blacktriangleright\!\!\blacktriangleright'$) does not have MAXISC exclusion rights. Then there exists a self-consistent scheme $\blacktriangleright\!\!\blacktriangleright^*$ that is larger than $\blacktriangleright\!\!\blacktriangleright'$. Consider the derived scheme $\tilde{\blacktriangleright\!\!\blacktriangleright}^*$ of the alternative model such that $i \tilde{\blacktriangleright\!\!\blacktriangleright}^* j$ if and only if $i \blacktriangleright\!\!\blacktriangleright^* j$ for any $i, j \in I$. That $\blacktriangleright\!\!\blacktriangleright^*$ is self-consistent implies that $\tilde{\blacktriangleright\!\!\blacktriangleright}^*$ is also self-consistent, and $\tilde{\blacktriangleright\!\!\blacktriangleright}^*$ is larger than $\tilde{\blacktriangleright\!\!\blacktriangleright}'$, which contradicts the assumption that $(\tilde{\blacktriangleright}', \tilde{\blacktriangleright\!\!\blacktriangleright}')$ has MAXISC exclusion rights.

By Claims (1) and (2), since the exclusion right system $(\tilde{\blacktriangleright}^p, \tilde{\blacktriangleright\!\!\blacktriangleright}^p)$ for the alternative model is proper, the derived exclusion right system $(\blacktriangleright, \blacktriangleright\!\!\blacktriangleright)$ is proper for the original model.

Part II. We show that if $\tilde{\mu}$ is the unique proper exclusion right core allocation of the alternative model $\langle I, \tilde{S}, \tilde{\succsim}, \tilde{\rhd}\rangle$, then $\mu = f(\tilde{\mu})$ is also a proper exclusion right core allocation of the original model $\langle I, S, Q, \succ, \rhd\rangle$. Suppose on the contrary that μ were not in the proper exclusion right core of the original model. That is, it would be blocked by a coalition $C \subseteq I$ such that there exists another allocation $\nu \in \mathcal{A}$ such that $\nu(i) \succ_i \mu(i)$ for all $i \in C$ and $\mu(j) \succ_j \nu(j)$ implies that there is an agent $i \in C$ such that $i \blacktriangleright\!\!\blacktriangleright_\mu j$. Since $i \blacktriangleright\!\!\blacktriangleright_\mu j$ implies $i \tilde{\blacktriangleright\!\!\blacktriangleright}_{\tilde{\mu}} j$, and for every $i \in C$, $\nu(i) \succ_i \mu(i)$ implies $\nu(i)^k \tilde{\succsim}_i \mu(i)^{k'}$ for any $k \in \{1, ..., q_{\nu(i)}\}$ and any $k' \in \{1, ..., q_{\mu(i)}\}$, allocation $\tilde{\mu}$ can be blocked by coalition C, contradicting that $\tilde{\mu}$ is a proper exclusion right core allocation. □

Proof of Theorem 5: Let $\tilde{\succsim}^o$ be the preferences of students under an ordering $o \in \mathcal{O}$. Let $\tilde{\mu}^o$ be the (unique) proper exclusion right core allocation of the alternative model $\langle I, \tilde{S}, \tilde{\succsim}^o, \tilde{\rhd}\rangle$. By Theorem 3, $\tilde{\mu}^o$ can be produced by the TTC algorithm denoted as $TTC(\tilde{\succsim}^o)$. Let I^o_k be the set of students who leave at step k of the algorithm $TTC(\tilde{\succsim}^o)$. Set $I^o_0 = I^{o'}_0 = \emptyset$. We show that $f(\tilde{\mu}^o) = f(\tilde{\mu}^{o'})$ holds true for any two orderings $o, o' \in \mathcal{O}$ by induction.

For any step $t \geq 1$, given the statement that for each step $1 \leq t' \leq t$, $I^o_{t'} = I^{o'}_{t'}$ and $f(\tilde{\mu}^o)(i) = f(\tilde{\mu}^{o'})(i)$ for every $i \in I^o_{t'}$, we show that the statement also holds true for step $t+1$. Clearly, every remaining school and the number of remaining seats of every remaining school are the same at the beginning of step $t+1$ in both $TTC(\tilde{\succsim}^o)$ and $TTC(\tilde{\succsim}^{o'})$. So, if there exists any student i who points to s_0 in $TTC(\tilde{\succsim}^o)$, then i also points to s_0 in $TTC(\tilde{\succsim}^{o'})$ so that the statement holds true in this case. Otherwise, each remaining student points to a seat of her most preferred school among remaining schools, and each remaining seat points to its top ranked remaining student in both

$TTC(\succsim^o)$ and $TTC(\succsim^{o'})$. Note that every student may point to different seats in the two orderings, but the seats belong to the same school, which is the most preferred one among remaining schools. Since every seat of the same school shares the same priority, all remaining seats of the same school will point to the same student. Therefore, the set of students involving in the cycles produced by $TTC(\succsim^o)$ are the same set of students in the cycles produced by $TTC(\succsim^{o'})$, i.e. $I^o_{t+1} = I^{o'}_{t+1}$. Since every student points to a seat of the same school in the two orderings, we have $f(\tilde{\mu}^o)(i) = f(\tilde{\mu}^{o'})(i)$ for every student $i \in I^o_{t+1}$. □

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