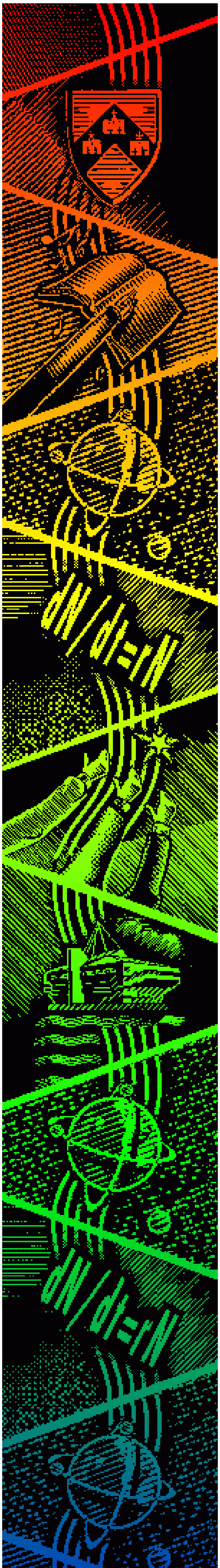

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Corporate Equilibrium Properties of a Centralized Objective Function GEI Model

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Abstract

We introduce an incomplete markets general equilibrium model with idiosyncratic risk, where production is financed via stock market, and where ownership structure endogenized. This model is a variation of Drèze (1974), Grossman and Hart (1979), and Magill and Quinzii (2002). The paper discusses two main corporate equilibrium properties. It shows that (i) the class of centralized objective functions introduces a further source of inefficiency into the organization of production, and (ii) the indeterminacy of corporate equilibria. (iii) It further shows the separation of the economic decisions of the agents.

1 Introduction

The seminal paper on production in incomplete markets dates back to Diamond [1]. He shows that under the assumption of multiplicative uncertainty the unique equilibrium in an one good, single agent economy is constraint efficient. Drèze [2], Grossman and Hart [3], and Magill and Quinzii [5] add more structure to this model and consider a two period model. Adding more structure to the economic model introduces a new economic phenomenon. Quinzii et al. [4] show that for this class of models equilibria are generally constrained inefficient.

This paper elaborates on this inefficiency property. It identifies a new source of inefficiency deriving from the utility dependent objective functions of the firms. The paper then studies the Modigliani and Miller theorem for the model introduced in this paper, and shows the separation of economic decisions of the agents. This is a first step towards a generalization of the decentralization theorem of the Arrow-Debreu model to incomplete markets.

Our model differs from the literature in three aspects. It considers technological uncertainty rather aggregate uncertainty, production is financed through the stock market, and ownership is endogenized. For this economic scenario, the paper shows that the organization of productive activities is generally inefficient. It identifies the organization of production as a further source of inefficiency. This result is a consequence of the utility dependent objective function of the firm. The paper shows that for this class of models financial policies are indeterminate, and that economic decisions can be separated. The later result is a weak form of the decentralization theorem of the Arrow-Debreu model.

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2 The Economic Model and Results

In the single agent reduced form model, the agent performs the role as a consumer and as a producer. As a consumer the agent buys stocks z and receives a proportion of the real value of the firm $\theta(z) = 1$ (in this case) in the next period in return. As a producer the agent issues the quantity of stocks b in order to finance a project. The return of financial investment the agent obtains as a consumer is denoted $R(\bar{y}, s)z$, and the dividend payoff the agent pays as a producer is denoted $R(\bar{y}, s)b$. The agent's $S + 1$ budget constraints are denoted

$$B_z = \left\{ \begin{array}{c} (x, z) \in \mathbb{R}^{S+1}_{++} \times \mathbb{R}_+ : \\ p(0)x(0) = p(0)\omega(0) - \theta(\bar{z})p(0)y(0) - qz + qb \\ p(s)x(s) = p(s)\omega(1) + \theta(\bar{z})p(s)y(s) + R(\bar{y}, s)\, z - R(\bar{y}, s)\, b \end{array} \right\}, \tag{1}$$

where $R(\bar{y}, s) = \frac{D(\bar{y},s)}{\hat{b}}$ is the dividend payoff per stock issued. Let $\xi = z + b$, then the agent's sequence of budget constraints can be rewritten as

$$B_\xi = \left\{ (x, \xi) \in \mathbb{R}^{S+1}_{++} \times \mathbb{R} : \begin{array}{c} p(0)x(0) = p(0)\omega(0) - \theta(\bar{z})p(0)y(0) - q\xi \\ p(s)x(s) = p(s)\omega(1) + \theta(\bar{z})p(s)y(s) + R(\bar{y}, s)\, \xi \end{array} \right\}, \tag{2}$$

where $p(0)y(0)$ denotes the investment costs in period one associated with revenue $p(s)y(s)$ in each state of nature $s \in \{1, ..., S\}$ in period two. In this model the firm's production set is $Y = \mathbb{R}^{S+1}$ if only one good in each state of nature is considered (otherwise $\mathbb{R}^{l(S+1)}$). Note that a price normalization implies that $p(0) = 1$, and $p(s) = 1$ in every $s \in \{1, ..., S\}$. The production set is described by a function $\Phi : \mathbb{R}_- \to \mathbb{R}^S_+$, where $Y = \{y \in \mathbb{R}^{S+1} : \Phi(y) \leq 0\}$. Standard assumptions for smooth technology sets apply. Ownership of the firm $\theta(.)$ is a function of quantity of stocks purchased as a consumer.

Definition 1 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$ for generic initial resources $\omega \in \Omega$ if following conditions are satisfied:*

$$\begin{array}{ll} (i) & \left(\bar{x}; \bar{\xi}, \bar{y}\right) \arg\max \{u(x) : x \in B_\xi\} \\ (ii) & \bar{\xi} = 0. \end{array} \tag{3}$$

Condition (ii) implies that the quantity of stocks that the consumer buys is equal to the quantity of stocks he issues as a producer. ξ denotes the net trade of stocks, where at equilibrium $\bar{\xi} = 0$ is satisfied. For the case that more than one consumption good is considered, $\bar{x}(0) = \omega(0) + \bar{y}(0)$, and $\bar{x}(s) = \omega(1) + \bar{y}(s)$ for all states of nature hold. The agent's optimization problem is to choose ξ and y such that utility of x is maximized.

Propositions (1), (2), and (3) state that in a single agent reduced form model, the utility maximization problem has a solution. The first two propositions show a first step towards modeling financial assets (on consumer side only), where ξ implying z and b implicitly contained in ξ, and for the case that the agent as a consumer takes financial policy of the firm b as given and chooses z to finance his preferred consumption bundle x. Proposition (3) shows the equivalence of these models.

Proposition 1 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $\left(\bar{x}, \bar{\xi}, \bar{y}\right)$ of the maximization problem (i), if and only if for generic initial resources $\omega \in \Omega$*

$$\bar{q} \text{ is a no-arbitrage price} \tag{4}$$

is satisfied.

Definition 2 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}, \bar{y})$ for for generic initial resources $\omega \in \Omega$ if following conditions are satisfied:*

$$\begin{array}{ll} \text{(i)} & (\bar{x}; \bar{z}, \bar{y}) \arg\max \{u(x) : (x; z, y) \in B_z\} \\ \text{(ii)} & \bar{z} + \hat{b} = 0 \end{array} \tag{5}$$

and $\bar{x}(0) = \omega(0) + \bar{y}(0)$, and $\bar{x}(s) = \omega(1) + \bar{y}(s)$ for all s hold for $l > 1$.

Proposition 2 *$(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}, \bar{y})$ of the maximization problem (i), if and only if for generic initial resources $\omega \in \Omega$*

$$\bar{q} \text{ is a no-arbitrage price} \tag{6}$$

is satisfied.

Proposition 3 *The reduced form model (1) and the reduced form model (2) are equivalent if*

$$\bar{\xi} = \bar{z} + \hat{b} \tag{7}$$

The reduced form model (2) is equivalent to the reduced form model (1) if

$$\bar{z} + \hat{b} = \bar{\xi}. \tag{8}$$

We now expand the reduced form model to an economic framework where decisions of the single agent are separated. This allows to introduce two separated optimizations problems, one for each role the agent plays. This example, although very simple, is non-trivial.

Suppose that the consumer assigns to the firm his own present value vector β. The objective of the agent as a producer is, given his own present value vector, to maximize the present value of streams of profits. This economic framework is sufficiently rich in structure in order to show the separation of activities of the agent as a consumer and as a producer. This is a weak form of the decentralization theorem of the Arrow-Debreu model.

Proposition 4 *$(\bar{p}, \bar{q})$ is a separated activities reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, (\bar{y}))$, for generic initial resources $\omega \in \Omega$, if and only if for $\bar{\beta}$ assigned to the objective function of the firm it satisfies:*

$$\begin{array}{ll} \text{(i)} & \left(\bar{x}, \bar{\xi}\right) \arg\max \{u(\bar{x}) : \bar{x} \in B_\xi\} \\ \text{(ii)} & (\bar{y}) \arg\max \left\{\bar{\beta}\bar{p}y : y \in Y\right\} \\ \text{(iii)} & \bar{\xi} = 0. \end{array} \tag{9}$$

Remark 1 *As a producer, the agent maximizes a present value problem not independent of information contained in the utility of the consumer. This makes sense in this one agent set up if one is willing to think of this model as an entrepreneurship model.*

Proposition (5) shows the inefficient organization of production of the reduced form model with separated activities of the agent. The degree of inefficiency introduced into the model depends on the consumer's present value vector.

Proposition 5 *The organization of production is generally (in)efficient for any assigned present value β to the objective function.*

The model is production efficient, if there does not exist a production plan $\hat{y} \neq y$ in Y such that $u(\hat{x}) > u(x)$. Alternatively, it is sufficient to expand the model to two consumers, and then, need to assign some arbitrarily determined average present value β_i to the objective function of the firm. It is easy to see that for any different present value vector assigned to the firm net activities change accordingly, hence $u(\hat{x}) \neq u(x)$.

The final result considers in the simplest form the irrelevance of financial policy theorem of Modigliani and Miller [6]. The theorem states that whatever financial policy a firm chooses, consumers can always undo this, leaving effects on real allocations unchanged. For that, we add more structure to the model and introduce an extensive form model of the firm, where financial policies are explicitly modeled. Denote the budget set of the consumer

$$B_z = \left\{(x, z) \in \mathbb{R}^{S+1}_{++} \times \mathbb{R} : px = p\omega + py + \Pi b + \Pi z\right\}, \tag{10}$$

where $\Pi = \begin{bmatrix} -q & \frac{D(1)}{b} & ,\ldots, & \frac{D(S)}{b} \end{bmatrix}$ is the financial payoff matrix (vector, here). $\frac{D(s)}{b}$ denotes the payoff per stock issued in a particular state of nature. As a consumer, the agent takes (p, q, b, y) as given and chooses z which finances his most preferred consumption bundle x. As a producer he takes (p, q, x, z) and present value vector β as given and chooses b and y such that present value profits are maximized. This is formally introduced in following definition.

Definition 3 $(\bar{p}, \bar{q})$ *is an extensive form equilibrium with associated equilibrium allocations* $(\bar{x}, \bar{z}), (\bar{y}, \bar{b})$, *for generic initial resources* $\omega \in \Omega$, *if following conditions are satisfied:*

$$\begin{array}{ll} (i) & (\bar{x}, \bar{z}) \arg\max \{u(x) : x \in B_z\} \\ (ii) & (\bar{y}, \bar{b}) \arg\max \left\{ \bar{\beta}\bar{p}y + \Pi b : \begin{array}{c} y \in Y \\ b \in \mathbb{R}_- \end{array} \right\} \\ (iii) & \bar{z} + \bar{b} = 0. \end{array} \tag{11}$$

Proposition (6) asserts that the precise nature of the producer's financial policy has no real effects on equilibrium allocations, provided it finances the producer's production plan. The result follows from showing the equivalence between the extensive form and the reduced form model where financial policies are not explicitly modeled. Two properties of this model make the proof work. (i) as a consumer and as a producer the agent has access to the same market subspace $\langle \Pi \rangle$, and (ii) a no-arbitrage condition $\beta\Pi = 0$ holds. Hence, financial polices do not affect the budget set of the consumer, nor the present value of future streams of profits generated by the producer. As a consumer, the single agent can always undo the financial activities taken as a producer. The value of the firm depends only on the production plan chosen by the producer, and not on its financial policy.

Proposition 6 *If $(\bar{p}, \bar{q})$ is an extensive form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z}), (\bar{y}, \bar{b})$, then $(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$ for generic initial resources $\omega \in \Omega$ where*

$$\bar{\xi} = \bar{z} + \bar{b} \tag{12}$$

If $(\bar{p}, \bar{q})$ is a reduced form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{\xi}, \bar{y})$, then $(\bar{p}, \bar{q})$ is an extensive form equilibrium with associated equilibrium allocations $(\bar{x}, \bar{z})$, $(\bar{y}, \bar{b})$ for generic initial resources $\omega \in \Omega$ and for all $(\bar{z}, \bar{b})$ satisfying

$$z + \bar{b} = \bar{\xi}. \tag{13}$$

3 Conclusion

The paper shows indeterminacy of corporate equilibria for a model with inefficient organization of production. It identifies the source of productive organizational inefficiency as a consequence of the utility dependent objective function of the firm. It further shows a preliminary result on the separation of economic activities of the agents.

The results suggest that the inefficient organization of production can be eliminated if it is possible to derive objective functions independent of the utilities of the shareholders. This is equivalent to generalizing the decentralization theorem of the Arrow-Debreu model to incomplete markets. This research is initiated in Stiefenhofer [7].

A Appendix: Mathematical Proofs

Proof 1 (Proposition 1) *Forming the Lagrangean*

$$\begin{aligned} L\left(\bar{x}, \bar{\xi}, \bar{y}, \bar{\lambda}, \bar{\mu}\right) = {} & u(x) - \lambda(0)\left[\bar{p}(0)x(0) - \bar{p}(0)\omega(0) + \bar{q}\xi - \theta(\bar{z})\bar{p}(0)y(0)\right] \\ & - \sum\nolimits_{s=1}^{S} \lambda(s)\left[\bar{p}(s)x(s) - \bar{p}(s)\omega(1) + \theta(\bar{z})\bar{p}(s)y(s) + R(\bar{y}, s)\xi\right] \\ & - \sum\nolimits_{s=0}^{S} \mu\left(s\right)\Phi(\bar{y}) \end{aligned} \tag{14}$$

The necessary and sufficient conditions for (x, ξ, y) to be a solution of L, are that there exists $\lambda \in \mathbb{R}^{S+1}_{++}$, and $\mu \in \mathbb{R}^{S+1}_{++}$ such that

$$\nabla L\left(\bar{x}, \bar{\xi}, \bar{y}, \bar{\lambda}, \bar{\mu}\right) \equiv 0$$

is satisfied. This is equivalent to

$$\begin{aligned} \nabla u(\bar{x}) &= \bar{\lambda}\bar{p} \\ \bar{q} &= \left(\frac{\sum_{s=1}^{S} \bar{\lambda}\left(s\right)}{\bar{\lambda}(0)}\right)\bar{p}(s)\bar{y}(s) \\ \bar{\mu}\nabla\Phi(\bar{y}) &= \bar{\lambda}\bar{p} \\ \bar{p}\bar{x} - \bar{p}\omega &= \theta(\bar{z})\bar{p}\bar{y} + \Pi(\bar{y}, \bar{p})\bar{\xi} \\ \Phi(\bar{y}) &= 0 \end{aligned} \tag{15}$$

where $\Pi = \begin{bmatrix} -q \\ p(s)y(s) \\ \vdots \\ p(S)y(S) \end{bmatrix}$. *Let* $\bar{\beta} = \left(\frac{\sum_{s=1}^{S}\bar{\lambda}(s)}{\bar{\lambda}(0)}\right)$. *Then* $\bar{q} = \sum_{s=1}^{S}\bar{\beta}(s)\bar{p}(s)\bar{y}(s)$. *It follows from the first order conditions that*

$$\bar{\beta}\bar{p} = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}) = \frac{\bar{\mu}}{\bar{\lambda}(0)}\nabla\Phi(\bar{y}) \tag{16}$$

Proof 2 (Proposition 2) *The necessary and sufficient conditions for* (x, z, y) *to be a solution of* L, *are that there exists* $\lambda \in \mathbb{R}^{S+1}_{++}$, *and* $\mu \in \mathbb{R}^{S+1}_{++}$ *such that*

$$\nabla L(\bar{x}, \bar{z}, \bar{y}, \bar{\lambda}, \bar{\mu}) \equiv 0$$

is satisfied. This is equivalent to

$$\begin{aligned} \nabla u(\bar{x}) &= \bar{\lambda}\bar{p} \\ \bar{q} &= \left(\frac{\sum_{s=1}^{S}\bar{\lambda}(s)}{\lambda(0)}\right)\bar{p}(s)\bar{y}(s) \\ \bar{\mu}\nabla\Phi(\bar{y}) &= \bar{\lambda}\bar{p} \\ \bar{p}\bar{x} - \bar{p}\omega &= \theta(\bar{z})\bar{p}\bar{y} + \Pi(\bar{p}, \bar{y})\bar{z} \\ \Phi(\bar{y}) &= 0 \end{aligned} \tag{17}$$

Proof 3 (Proposition 3) *From (1) have* $\bar{\xi} = 0$, *and from (2) have* $\bar{z} + \hat{b} = 0$. *The equivalence follows from* $\bar{\xi} = \bar{z} + \hat{b} = 0$.

Proof 4 (Proposition 4) *Suppose that the agent assigns* $\bar{\beta}$ *to the producer. It remains to show that*

$$\max_{\bar{y}}\left\{\bar{\beta}\bar{p}y : y \in Y\right\} \tag{18}$$

is well defined. Since the first order conditions are such that there exists $\mu \in \mathbb{R}^{S+1}_{++}$. *From*

$$\nabla L(\bar{y}) \equiv 0 \tag{19}$$

have

$$\bar{\beta}\bar{p} = \bar{\nu}\nabla\Phi(\bar{y}) \tag{20}$$

it follows that

$$\bar{\beta}\bar{p} = \bar{\nu}\nabla\Phi(\bar{y}) \Longleftrightarrow \frac{\bar{\mu}}{\bar{\lambda}(0)}\nabla\Phi(\bar{y}) = \frac{1}{\bar{\lambda}(0)}\nabla u(\bar{x}) = \bar{\beta}\bar{p} \tag{21}$$

Proof 5 (Proposition 5) *The source of inefficiency comes from the no-arbitrage condition,* $\beta\Pi = 0$. *This equation is indeterminate for the case that* $S > n$. *Therefore for any* $\hat{\beta} \neq \beta$ *assigned to the firm it follows that* $\bar{y}|_{\hat{\beta}} \neq \bar{y}|_{\beta}$ *in* Y *since*

$$\max_{\bar{y}|_{\hat{\beta}}}\left\{\hat{\beta}\bar{p}y : y \in Y\right\} \quad \neq \quad \max_{\bar{y}|_{\beta}}\left\{\beta\bar{p}y : y \in Y\right\} \tag{22}$$

Proof 6 (Proposition 6) *(1) show that* $(\bar{x}, \bar{\xi}, \bar{y})$ *satisfies the first order conditions so that (i) in definition of a reduced form equilibrium is satisfied. The first order conditions are*

$$\bar{p}\bar{x} - \bar{p}\bar{\omega} = \bar{p}\bar{y} + \Pi\bar{b} + \Pi\bar{z}, \text{ and } \bar{\beta}\Pi = 0 \tag{23}$$

which is equivalent to

$$\bar{p}\bar{x} - \bar{p}\bar{\omega} = \bar{p}\bar{y} + \Pi\bar{\xi}, \text{ and } \bar{\beta}\Pi = 0 \tag{24}$$

since $\bar{\xi} = \bar{z} + \bar{b}$ *holds, so that first order condition (above) holds. Next, show what the no-arbitrage condition implies for the firm for all* $(\bar{y}, \bar{b})$*, the present value of the firm to the producer reduces to*

$$\bar{\beta}\bar{p}\bar{y} = \bar{\beta}\bar{p}\bar{y} + \bar{\beta}\Pi\bar{b} = \bar{\beta}\bar{p}\bar{y} \tag{25}$$

Thus the producer's problem in the extensive form equilibrium definition is equivalent to

$$(\bar{y}) \arg\max \left\{\bar{\beta}\bar{p}y : y \in Y\right\} \tag{26}$$

for which the first order conditions are given (above). The last step is to recall that the market clearing condition $\bar{\xi} = \bar{z} + \bar{b} = 0$ *holds, and from which the result follows.*
(2) show that if $(\bar{x}, \bar{\xi}, \bar{y})$ *is a solution to the reduced form problem, then the first order conditions(above) are satisfied. This implies that, for any* $b \in \mathbb{R}$

$$\left(\bar{y}, \bar{b}\right) \arg\max \left\{\bar{\beta}\bar{p}y + \Pi\bar{b} : (y; b) \in Y \times \mathbb{R}\right\} \tag{27}$$

since by no-arbitrage condition $\bar{\beta}\Pi = 0$*. Therefore, can pick any* $b \in \mathbb{R}$*, and define*

$$z = \bar{\xi} - \bar{b} \tag{28}$$

then the first order condition of extensive form equilibrium is satisfied by $(\bar{x}, z)$*, and thus* $(\bar{x}, z)$ *is a solution of the extensive form equilibrium, since* $(\bar{y}, \bar{b})$ *is a solution of the extensive form equilibrium, and the result follows from* $0 = \bar{\xi} = z + \bar{b}$*.*

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