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Abstract

This paper investigates the uniqueness of subgame perfect (SP) payoffs in a sequential bargaining game. Players are completely informed and the surplus to be allocated follows a geometric Brownian motion. This bargaining problem has not been analysed exhaustively in a stochastic environment. The aim of this paper is to provide a technique to identify the subgame perfect equilibria, i.e. the timing of the agreement and the SP payoff at which agreement occurs. Even though the main focus is on the uniqueness of the equilibrium, we investigate other features of the equilibrium, such as the Pareto efficiency of the outcome and the relation with the Nash axiomatic approach.

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Introduction

Since the seminal work of Rubinstein (1982) the basic two-player game has been generalized, inspiring a vast literature covering theoretical issues as well as economic and finance applications.

The extension of the Rubinstein model to the stochastic case is a new research area[1]. Even though the theoretical and applied works in the field of non cooperative bargaining have covered a broad variety of issues, the extension of the basic model (with players completely informed about their preferences) to the stochastic environment seems to have made some inroads just recently.

There are only few papers on sequential games where a stochastic process drives the order of the moves and/or the "pie" to be shared. In particular, we will focus on two papers investigating the Rubinstein alternating-offer model in a stochastic environment.

Merlo and Wilson (1995) derive necessary and sufficient conditions for existence and uniqueness of subgame perfect and stationary subgame perfect payoffs in a generalised setting. There are 'n' players, proposers alternate randomly and/or the pie moves stochastically with time. In particular, the uncertainty does not take any specific form, the underlying stochastic process is left unspecified such that their conclusions on existence and uniqueness of subgame perfect (SP) payoff can apply to a multiplicity of stochastic processes. However, given the generality of the underlying process, a number of perverse results are possible and a conclusive result on issues like efficiency of the equilibria does not seem feasible. This makes quite hard to apply their theoretical result as a technical tool to solve a bargaining situation in a more specific stochastic framework.

A more specific application is given in Cripps (1997) who analyses a Rubinstein alternating bargaining situation where the pie moves accordingly to a geometric Brownian motion. The focus is from an applied perspective and the aim is to solve a long and short-term wage negotiation between a firm and a labour union. The equilibrium found is unique in the 'C2' class of functions, however a general characterisation of the uniqueness is a technical issue that is beyond the scope of his paper (Cripps, 1997 p. 536).

The main purpose of this essay is to refine and extend the model developed by Cripps. First, the essential technical refinement is the characterisation of the equilibrium in terms of uniqueness. In doing this, drawing on the same bargaining situation as in Cripps, we provide the characterisation of SP equilibrium payoffs and the technical steps yielding the unique SP equilibrium of the model. In particular, the focus is on stationary

[1]For a discussion on this issue see Muthoo (1999).

subgame perfect (SSP) payoffs because in a two-player game the SSP payoff is unique if and only if SP payoff is unique[2]. Second, the current paper extends Cripps' analysis by investigating the Pareto efficiency of the bargaining outcome. Third, we analyse the equilibrium outcome in the alternative situation where players can bargain cooperatively in a Nash demand game. Finally we compare the equilibrium outcomes of the two alternative bargaining situations – non-cooperative Rubinstein framework and cooperative Nash bargaining. This comparison highlights the delay in the investment decision due to the lack of a cooperative environment.

The paper is structured as follows. In section 1, we briefly describe the bargaining procedure in the basic deterministic case and in section 1.1, the specific bargaining problem – negotiation over wage contracts – when the pie is stochastic[3]. In section 2, we provide a characterisation of the SSP payoffs by defining an upper and a lower bound respectively to the supremum and the infimum of the SSP outcomes. The main question concerns the uniqueness of the subgame perfect equilibrium payoff. In section 3, the existence of a unique SSP outcome is proved by shrinking the set of SSP payoffs to a single point[4]. In this section the solution depends on the time interval elapsing between an offer and counteroffer, however in the rest of the paper continuous time is assumed. The main result following from this section is that there exists a unique SSP partition of the pie at which players agree immediately if the state variable starts sufficiently high (above a constant threshold level). The level of the stochastic cash flows that triggers such an agreement is assumed to exist in the sense of being finite.

In section 4, we solve for the unknown threshold level that triggers an agreement. Then we investigate the timing of the agreement for any initial level of the state variable in order to detect if any earlier SSP agreement is feasible when the state variable starts sufficiently low. One may notice how this technique closely resembles a kind of backward induction even though the time horizon is infinite.

In subsection 4.1, we draw the main conclusions on timing and efficiency of the SSP outcome. Whether the SSP outcome is Pareto efficient is a minor, though relevant, question. In the basic model the SSP outcome is efficient and an agreement is reached at the first round of the game. Alternatively, in a stochastic framework, agreement could be

[2]This follows the fact that in a two player game extremal SP payoffs are also SSP payoffs. See Merlo and Wilson (1995, p.391).

[3]The notation here is the same as in Cripps so that a comparison with his result may be straightforward to the reader.

[4]We will do this by using the upper and lower bounds from section 2.

delayed because players perceive that by waiting they receive an improved payoff. In such a case, as concluded by Merlo and Wilson (1995, p.385), it cannot be that any feasible state contingent outcome can be agreed upon in the current state and in such a case one cannot guarantee that a SSP equilibrium is Pareto optimal. However, in the current paper, it is straightforward to show that if there is a delay (because the state variable starts sufficiently low) the SSP outcome at agreement is still Pareto efficient.

Before concluding, in section 5 we briefly introduce the Nash bargaining problem in order to make a comparison with the Rubinstein sequential bargaining. In the Nash framework, the Pareto optimality condition is defined more explicitly because the efficiency is one of the axioms the Nash solution must satisfy. We show then how the Nash and the SSP outcomes satisfy the efficiency condition. This relation, through the Pareto optimality, facilitates a comparison between the two bargaining situations. This comparison is quite useful if one notices that the efficient SSP outcome is the implicit solution of a polynomial equation of degree $\lambda > 1$. In contrast, the Nash bargaining solution is explicitly defined and easy to calculate. From this comparison one can conclude that, with respect to the Nash outcome, the efficient SSP agreement occurs at a higher level of the the state variable and, hence, the lack of cooperation delays agreements and investment decisions.

1 Alternating-Offer Game: Bargaining Procedure

In this section, we will briefly go through the basic deterministic model of sequential bargaining. We will do this by summarising the bargaining procedure and the outcome without dealing with other features of the solution such as subgame perfection, uniqueness and efficiency. These features will be described extensively in the following section for the stochastic case.

The main features of this type of bargaining are (i) the bargaining has an infinite horizon and (ii) it is costly.

Two players, say 1 and 2, have the opportunity to share a pie of size 1. They alternate between making an offer at discrete points in time. To allow for serious offers, assume the two agents are impatient with discount factors $0 < D_i < 1$, $i = 1, 2$. The impatience then represents the cost of bargaining. Assume player 1 starts the negotiation. He proposes at time zero a partition of the pie $x = (x_1, x_2)$ with $x_1 + x_2 = 1$. Player 2 can either accept this proposal, in which case the game ends with the implementation of the partition x, or reject it. If he rejects, the game moves on by a time period Δ, with player 2 making a proposal $y = (y_1, y_2)$ with $y_1 + y_2 = 1$. If the offer is rejected the game goes on with a sequence of alternating offers until an proposal is accepted. If the bargaining never reaches an agreement both players receive a zero payoff as the discounted pie shrinks to zero.

This game admits a unique equilibrium payoff, namely:

$$x = \left(\frac{1 - D_2}{1 - D_1 D_2}, \frac{(1 - D_1) D_2}{1 - D_1 D_2}\right), \qquad y = \left(\frac{(1 - D_2) D_1}{1 - D_1 D_2}, \frac{(1 - D_1)}{1 - D_1 D_2}\right)$$

and the agreement is reached at the first round[5].

1.1 Alternating-Offer Over a Stochastic Pie

The purpose of this section is first to describe the features of the wage negotiation (and hence the features of the wage-contract), then we will describe the bargaining procedure.

An entrepreneur has the opportunity to invest in a project generating an instantaneous cash flow, P_t, which follows a geometric Brownian motion

$$dP_t = \mu P_t dt + \sigma P_t dz_t,$$

where μ and σ are the drift and the volatility of the process respectively. In order to invest, the entrepreneur pays an initial fixed cost, K, and hires a labourer who earns an instantaneous constant wage, w, until the production stops. We assume that there is monopoly on the demand and supply sides of the labour market, therefore in order to invest, the manager and the labourer (or a labour union) have to agree on a long-term fixed wage contract, w.

The union and the manager are risk neutral and have different time preferences over payoffs described respectively by discount factors $e^{-\gamma t}$ and $e^{-\delta t}$. The time preferences are such that δ and γ are both greater than μ.

Bargaining structure

Agents play an alternating offer game over wage contracts. If the entrepreneur has the right to start the game, then the bargaining runs as follows.

Step 1

At time zero the manager observes the initial level of the cash flow, P_t. At this level of the state variable his (own evaluation of the) expected discounted cash flows, minus

[5] See Binmore (1987) for a refinement of the Rubinstein result in terms of subgame perfection.

the investment cost K, is

$$\int_t^{\infty} P_s e^{-\gamma(s-t)} ds - K = \frac{P_t}{\gamma - \mu} - K,$$

which represents the value of the 'pie' (the net value of the investment) to the eyes of the entrepreneur.

Once the value of the pie is known, the manager makes an offer to the union. If the union accepts this offer the manager obtains a share of the pie, say $\alpha_\Delta(P_t)$ and the union receives the residual share, say $\tilde{\beta}_\Delta(P_t)$, where

$$\tilde{\beta}_\Delta(P_t) = \left[\frac{P_t}{\gamma - \mu} - K - \alpha_\Delta(P_t)\right]\frac{\delta}{\gamma}. \tag{1}$$

We will use the tilde to refer to one player's payoff under the offer of the other player.

Under this agreement the investment starts and the instantaneous constant wage, w, is determined because the union's share of the pie is equal to w/δ.

Step 2

If the union rejects the manager's offer, at time $t + \Delta$ (one period ahead), the union makes a proposal. The union observes the state variable $P_{t+\Delta}$, calculates the value of the pie, which is given by

$$\left[\frac{P_{t+\Delta}}{\gamma - \mu} - K\right]\frac{\gamma}{\delta}, \tag{2}$$

and makes an offer to the manager. If the offer is accepted the game ends and investment starts with payoffs to the manager and the union respectively equal to

$$\tilde{\alpha}_\Delta(P_{t+\Delta}) = \frac{P_{t+\Delta}}{\gamma - \mu} - K - \beta_\Delta(P_{t+\Delta})\frac{\gamma}{\delta}, \qquad \beta_\Delta(P_{t+\Delta}). \tag{3}$$

If the offer is rejected, the game continues with steps 1 and 2 repeating until an offer is accepted. If the bargaining never reaches an agreement, both players receive zero[6].

2 Characterisation of Subgame Perfect Equilibrium Payoffs

In this section, we briefly characterise the set of stationary subgame perfect (SSP) payoffs by defining its supremum and infimum. In the following section, this characterisation

[6]Because the time preferences are greater than the drift of the process, the expected discounted value of the pie shrinks to zero as in the deterministic case.

allows us to investigate uniqueness by checking whether the supremum and the infimum are equal. In such a case the SSP set shrinks to a single point.

Let $A(P_t)$ and $a(P_t)$ ($B(P_t)$ and $b(P_t)$) denote respectively the supremum and the infimum of the share for the manager, player M, (the union, player U) in any perfect equilibrium of the game when he (she) is the proposer. And let $\tilde{A}(P_t)$ and $\tilde{a}(P_t)$ ($\tilde{B}(P_t)$ and $\tilde{b}(P_t)$) be the supremum and the infimum when M (U) is the responder.

Consider the subgame beginning at time t with an offer made by M. In this subgame any offer made by M which gives U a share greater than $E_t B(P_{t+\Delta})e^{-\delta\Delta}$ will be accepted by U (being $B(P_{t+\Delta})$ the supremum of the share to U at his turn one period ahead). Then M can guarantee at least a share

$$\frac{P_t}{\gamma - \mu} - K - E_t B(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma} \leq a(P_t) \tag{4}$$

and U can obtain at most

$$E_t B(P_{t+\Delta})e^{-\delta\Delta} \geq \tilde{B}(P_t). \tag{5}$$

Now consider an offer made by M which gives U a share less than $E_t b(P_{t+\Delta})e^{-\delta\Delta}$. Any proposal like this will be rejected by U (being $b(P_{t+\Delta})$ the infimum of the share U earns next stage). Therefore

$$\tilde{b}(P_t) \geq E_t b(P_{t+\Delta})e^{-\delta\Delta} \tag{6}$$

and the best M can obtain by making an acceptable proposal is $\frac{P_t}{\gamma-\mu} - K - E_t b(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma}$. If U rejects this offer then M, as responder next stage, will receive at most $E_t\tilde{A}(P_{t+\Delta})e^{-\gamma\Delta}$. This can be written as

$$A(P_t) \leq \max\left\{\frac{P_t}{\gamma - \mu} - K - E_t b(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma}, E_t\tilde{A}(P_{t+\Delta})e^{-\gamma\Delta}\right\}. \tag{7}$$

Combining 4 and 7 with the definition of supremum and infimum gives the lower and upper bounds to the set of SSP payoffs to M, that is:

$$\begin{aligned} &\frac{P_t}{\gamma - \mu} - K - E_t B(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma} \leq a(P_t) \leq A(P_t) \\ &\leq \max\left\{\frac{P_t}{\gamma - \mu} - K - E_t b(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma}, E_t\tilde{A}(P_{t+\Delta})e^{-\gamma\Delta}\right\}. \end{aligned} \tag{8}$$

Similarly 5 and 6 yield bounds for the set of SSP payoffs to U, as responder, which are given by

$$E_t b(P_{t+\Delta})e^{-\delta\Delta} \leq \tilde{b}(P_t) \leq \tilde{B}(P_t) \leq E_t B(P_{t+\Delta})e^{-\delta\Delta}. \tag{9}$$

Repeating the same argument for the subgame beginning at time $t+\Delta$, with a proposal made by U (where U offers more than $E_{t+\Delta}A(P_{t+2\Delta})e^{-\gamma\Delta}$ or less than $E_{t+\Delta}a(P_{t+2\Delta})e^{-\gamma\Delta}$), yields bounds to the set of SSP to U and M (as responder) respectively

$$\begin{aligned} &\left[\frac{P_{t+\Delta}}{\gamma-\mu} - K - E_{t+\Delta}A(P_{t+2\Delta})e^{-\gamma\Delta}\right]\frac{\delta}{\gamma} \leq b(P_{t+\Delta}) \leq B(P_{t+\Delta}) \\ \leq \max &\left\{\left[\frac{P_{t+\Delta}}{\gamma-\mu} - K - E_{t+\Delta}a(P_{t+2\Delta})e^{-\gamma\Delta}\right]\frac{\delta}{\gamma}, E_{t+\Delta}\tilde{B}(P_{t+2\Delta})e^{-\delta\Delta}\right\}, \end{aligned} \tag{10}$$

and

$$E_{t+\Delta}a(P_{t+2\Delta})e^{-\gamma\Delta} \leq \tilde{a}(P_{t+\Delta}) \leq \tilde{A}(P_{t+\Delta}) \leq E_{t+\Delta}A(P_{t+2\Delta})e^{-\gamma\Delta}. \tag{11}$$

3 Uniqueness of SSP Payoffs

Definition 1. Optimal stopping time (OST). Corresponding to

i) any SSP payoff

$$\bar{\alpha}(P_t) = \begin{cases} A(P_t) & \text{for M proposing at t,} \\ \tilde{A}(P_t) & \text{for U proposing at t.} \end{cases}$$

there is an OST, $\tau_{\bar{\alpha}} = \{\min n \geq 0 : \bar{\alpha}(P_n) \geq \sup_{j>n} E_n\bar{\alpha}(P_j)e^{-\gamma(j-n)}\}$.

ii) any SSP payoff

$$\bar{\beta}(P_t) = \begin{cases} B(P_t) & \text{for U proposing at t,} \\ \tilde{B}(P_t) & \text{for M proposing at t.} \end{cases}$$

there is an OST, $\tau_{\bar{\beta}} = \{\min n \geq 0 : \bar{\beta}(P_n) \geq \sup_{j>n} E_n\bar{\beta}(P_j)e^{-\delta(j-n)}\}$.

It follows that extremal SSP payoffs $(\bar{\alpha}(P_t), \bar{\beta}(P_t))$ would be proposed or accepted at any time $t \geq \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$ because neither player, by the definition of OST, could improve his best outcome.

Note that the assumption on the existence of OST yielding extremal payoffs just excludes the trivial solution[7] (0,0).

We will proceed as follows. First one can use definition 1 to calculate the max in 8 and 10 when the initial level of the state variable P_t is such that $t \geq \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$. Then the

[7] If an optimal stopping rule does not exist, in the sense that either $\tau_{\bar{\alpha}}$ or $\tau_{\bar{\beta}}$ is infinite, then $\max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\} = \infty$. This implies that an agreement (yielding extremal payoffs) never occurs and $(\bar{\alpha}(P_t) = 0, \bar{\beta}(P_t) = 0)$. Then the supremum to M and U is zero and the set of SSP payoffs would only contain the disagreement point.

upper bound to M and U can be solved recursively. If extremal payoffs like $(\frac{P_t}{\gamma-\mu}-K,0)$ can be ruled out by the convergence of the upper bound, then there is a unique SSP payoff at which an agreement occurs at the first round of the game.

Combining the last inequality in 11 with definition 1 i), where $j-n$ can be set equal to Δ, yields $\tilde{A}(P_{t+\Delta}) = E_{t+\Delta}A(P_{t+2\Delta})e^{-\gamma\Delta}$ for any $t > \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$. Then 7 becomes

$$A(P_t) \leq \max\left\{\frac{P_t}{\gamma-\mu} - K - E_t b(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma}, E_t E_{t+\Delta}A(P_{t+2\Delta})e^{-\gamma 2\Delta}\right\}. \tag{12}$$

Also, by definition 1 i) (with $j-n=2\Delta$) it must be $A(P_t) \geq E_t A(P_{t+2\Delta})e^{-\gamma 2\Delta}$ which combined with 12 yields

$$E_t A(P_{t+2\Delta})e^{-\gamma 2\Delta} \leq A(P_t) \leq \frac{P_t}{\gamma-\mu} - K - E_t b(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma}. \tag{13}$$

This can be rearranged (by the first inequality in 10) as

$$\begin{aligned} E_t A(P_{t+2\Delta})e^{-\gamma 2\Delta} &\leq A(P_t) \\ &\leq \frac{P_t}{\gamma-\mu}(1-e^{-(\delta-\mu)\Delta}) - K(1-e^{-\delta\Delta}) + E_t A(P_{t+2\Delta})e^{-(\delta+\gamma)\Delta}, \end{aligned} \tag{14}$$

where the right hand side can be solved recursively. Using for brevity the notation $c_1 = \frac{1-e^{-(\delta-\mu)\Delta}}{\gamma-\mu}$ and $c_2 = 1-e^{-\delta\Delta}$ at the nth iteration, 14 expands as follows:

$$\begin{aligned} A(P_t) &\leq P_t c_1 - K c_2 + E_t A(P_{t+2\Delta})e^{-(\delta+\gamma)\Delta} \\ &\leq P_t c_1 \sum_{i=0}^{n-1} e^{-i(\delta+\gamma-2\mu)\Delta} - K c_2 \sum_{i=0}^{n-1} e^{-i(\delta+\gamma)\Delta} + E_t A(P_{t+n2\Delta})e^{-n(\delta+\gamma)\Delta}. \end{aligned} \tag{15}$$

As $n \to \infty$, then $E_t A(P_{t+n2\Delta})e^{-n(\delta+\gamma)\Delta}$ converges to 0. This follows the fact that $A(P_{t+n2\Delta}) \leq \frac{P_{t+2n\Delta}}{\gamma-\mu} - K$ and then:

$$\begin{aligned} \lim_{n\to\infty} E_t A(P_{t+n2\Delta})e^{-n(\delta+\gamma)\Delta} &\leq \lim_{n\to\infty} E_t \frac{P_{t+2n\Delta}}{\gamma-\mu}e^{-n(\delta+\gamma)\Delta} - Ke^{-n(\delta+\gamma)\Delta} \\ &\leq \lim_{n\to\infty} \frac{P_t}{\gamma-\mu}e^{-n(\delta+\gamma-2\mu)\Delta} - Ke^{-n(\delta+\gamma)\Delta} \\ &= 0. \end{aligned} \tag{16}$$

Therefore as $n \to \infty$, the upper bound to $A(P_t)$, with $t \geq \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$, converges to

$$\frac{P_t c_1}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{K c_2}{1-e^{-(\delta+\gamma)\delta}} \geq A(P_t). \tag{17}$$

Regarding U, a similar straightforward calculation at time $t+\Delta \geq \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$ yields the upper bound to U, i.e.

$$\left(\frac{P_{t+\Delta}c_3}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{Kc_4}{1-e^{-(\delta+\gamma)\delta}}\right)\frac{\gamma}{\delta} \geq B(P_{t+\Delta}), \tag{18}$$

where $c_3 = \frac{1-e^{-(\gamma-\mu)\Delta}}{\gamma-\mu}$ and $c_3 = 1-e^{-\gamma\Delta}$.

It is now straightforward to show that the lower bound and upper bound are the same. In fact, combining 4 and 18 gives

$$\begin{aligned} a(P_t) &\geq \frac{P_t}{\gamma-\mu} - K - E_t B(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma} \\ &\geq \frac{P_t}{\gamma-\mu} - K - E_t\left(\frac{P_{t+\Delta}c_3}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{Kc_4}{1-e^{-(\delta+\gamma)\delta}}\right)e^{-\delta\Delta} \\ &= \frac{P_t c_1}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{Kc_2}{1-e^{-(\delta+\gamma)\delta}}, \end{aligned} \tag{19}$$

which implies

$$A(P_t) = a(P_t) = \frac{P_t c_1}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{Kc_2}{1-e^{-(\delta+\gamma)\delta}}. \tag{20}$$

Moreover combining 10, 18 and 20 yields

$$B(P_{t+\Delta}) = b(P_{t+\Delta}) = \left(\frac{P_{t+\Delta}c_3}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{Kc_4}{1-e^{-(\delta+\gamma)\delta}}\right)\frac{\gamma}{\delta}. \tag{21}$$

Therefore, one concludes that at any level of cash flow, P_t (with $t \geq \max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$), M (U) always proposes an allocation yielding $\alpha_\Delta(P_t) = A(P_t) = a(P_t)$ ($\beta_\Delta(P_t) = B(P_t) = b(P_t)$) when he (she) is the proposer at t, and accepts a share $\tilde{\alpha}_\Delta(P_t) = E_t\alpha_\Delta(P_{t+\Delta})e^{-\gamma\Delta}$ ($\tilde{\beta}_\Delta(P_t) = E_t\beta_\Delta(P_{t+\Delta})e^{-\delta\Delta}$) when U (M) is the proposer at t[8]. Moreover, as the SSP payoffs proposed (by M/U) and accepted (by U/M) sum to the 'pie' (evaluated by M), i.e.

$$\begin{aligned} E_t\beta_\Delta(P_{t+\Delta})e^{-\delta\Delta}\frac{\delta}{\gamma} + \alpha_\Delta(P_t) &= \frac{P_t}{\gamma-\mu} - K, \\ E_t\alpha_\Delta(P_{t+\Delta})e^{-\gamma\Delta} + \beta_\Delta(P_t)\frac{\delta}{\gamma} &= \frac{P_t}{\gamma-\mu} - K, \end{aligned}$$

it follows that an agreement occurs at the first round of the game with (SSP) equilibrium payoffs to M and U equal to $(\alpha_\Delta(P_t), E_t\beta_\Delta(P_{t+\Delta})e^{-\delta\Delta})$, when M is the proposer at t, and equal to $(E_t\alpha_\Delta(P_{t+\Delta})e^{-\gamma\Delta}, \beta_\Delta(P_t))$, when U makes a proposal.

[8]The subscript Δ is to remind us that we are in discrete time and the outcome depends on Δ.

4 Timing of the agreement

The objective of this section is to solve the optimal stopping time problem for both players in order to find $\max\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\}$.

As this does not change the result about uniqueness but makes the calculation easier, we will proceed by analysing the continuous-time case. As $\Delta \to 0$

$$\begin{aligned}\alpha_\Delta(P_t) &= \frac{P_t c_1}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{K c_2}{1-e^{-(\delta+\gamma)\delta}} \\ &\to \frac{P_t}{\gamma-\mu}\frac{\delta-\mu}{\delta+\gamma-2\mu} - K\frac{\delta}{\delta+\gamma} = \alpha(P_t)\end{aligned} \tag{22}$$

and

$$\begin{aligned}\beta_\Delta(P_t) &= \left(\frac{P_{t+\Delta} c_3}{1-e^{-(\gamma+\delta-2\mu)\Delta}} - \frac{K c_4}{1-e^{-(\delta+\gamma)\delta}}\right)\frac{\gamma}{\delta} \\ &\to \left(\frac{P_t}{\delta+\gamma-2\mu} - K\frac{\gamma}{\delta+\gamma}\right)\frac{\gamma}{\delta} = \beta(P_t).\end{aligned} \tag{23}$$

Moreover, by definition 1, the optimal stopping problems to be solved can be written as

$$E_t\alpha(P_{\tau_\alpha})e^{-(\tau_\alpha-t)\gamma} = \sup_\tau E_t\alpha(P_\tau)e^{-(\tau-t)\gamma} \tag{24}$$

and

$$E_t\beta(P_{\tau_\beta})e^{-(\tau_\beta-t)\delta} = \sup_\tau E_t\beta(P_\tau)e^{-(\tau-t)\delta} \tag{25}$$

for M and U respectively.

First, we will solve a general stopping problem associated with a geometric Brownian motion.

Proposition 1. *Assuming the time preference, r, is greater than the drift of the process, μ, if τ is a stopping time, then the problem $E_t g(P_\tau)e^{-(\tau-t)r}$ has solution $g(P^*)\left(\frac{P_t}{P^*}\right)^\lambda$ where $\lambda = \left(-(\mu-\sigma^2) \pm \sqrt{(\mu-\sigma^2)^2+2\sigma^2 r}\right)/\sigma^2$.*

Proof As $E_t g(P_\tau)e^{-(\tau-t)r} = E_t E_\tau g(P_\tau)e^{-(\tau-t)r} = g(P^*)E_t e^{-(\tau-t)r}$ then the problem reduces to solving the term $E_t e^{-(\tau-t)r}$.

We will use a martingale theorem for stopping times[9]. If τ is a stopping time, then we can define a martingale $E_t e^{-\frac{\xi^2}{2}(\tau-t)+\xi\Delta Z(\tau-t)} = 1$ for any $\xi \in \mathbb{R}$. This property can be

[9]See Ross (1996), pp. 381-302.

combined with the well known result, $\ln P_\tau/P_t = (\mu-\sigma^2)(\tau-t)+\sigma\Delta Z(\tau-t)$, as follows

$$
\begin{aligned}
E_t e^{-\frac{\xi^2}{2}(\tau-t)+\xi\Delta Z(\tau-t)} &= E_t e^{-\frac{\xi^2}{2}(\tau-t)+[\ln P^*/P_t-(\mu-\sigma^2/2)(\tau-t)]\xi/\sigma} \\
&= E_t\left(\frac{P^*}{P_t}\right)^{\xi/\sigma} e^{-\frac{\xi^2}{2}(\tau-t)-\frac{\xi}{\sigma}(\mu-\sigma^2/2)(\tau-t)} \\
&= 1.
\end{aligned}
$$

From the last equality then $E_t e^{-[\frac{\xi^2}{2}+\frac{\xi}{\sigma}(\mu-\sigma^2/2)](\tau-t)} = \left(\frac{P_t}{P^*}\right)^{\xi/\sigma}$. Then this result implies that $E_t e^{-(\tau-t)r} = \left(\frac{P_t}{P^*}\right)^{\xi/\sigma}$ where ξ solves $E_t e^{-[\frac{\xi^2}{2}+\frac{\xi}{\sigma}(\mu-\sigma^2/2)](\tau-t)} = E_t e^{-(\tau-t)r}$. It follows that ξ is the solution to $[\frac{\xi^2}{2}+\frac{\xi}{\sigma}(\mu-\sigma^2/2)] = r$; this gives the two roots $\xi = \left(-(\mu-\sigma^2)\pm\sqrt{(\mu-\sigma^2)^2+2\sigma^2 r)}\right)/\sigma$. Notice that $\xi_1 > 0$ and $\xi_2 < 0$ being $r > 0$. Observe that $E_t e^{-(\tau-t)r} \leq 1$ for $t \leq \tau$ so it must be $P_t \leq P^*$ for $\xi > 0$ and $P_t \geq P^*$ otherwise[10]. Let $\xi/\sigma = \lambda$ then $E_t e^{-(\tau-t)r} = \left(\frac{P_t}{P^*}\right)^{\lambda}$. ■

With this result the problem $\sup_\tau E_t g(P_\tau)e^{-(\tau-t)r}$ can be rearranged as $\max_{P^*} g(P^*)\left(\frac{P_t}{P^*}\right)^{\lambda}$ with first and second order conditions:

$$
\begin{aligned}
&foc \rightarrow [g'(P^*)P^{*^{-\lambda}} - \lambda g(P^*)P^{*^{-\lambda-1}}]P_t^{\lambda} = 0 \\
&soc \rightarrow [g''(P^*)P^{*^{-\lambda}} - \lambda g'(P^*)P^{*^{-\lambda-1}} - \lambda g'(P^*)P^{*^{-\lambda-1}} + \lambda(\lambda+1)g(P^*)P^{*^{-\lambda-2}}]P_t^{\lambda} < 0.
\end{aligned}
$$

Notice that in our case $\alpha(P_t)$ and $\beta(P_t)$ are straight lines with positive slope and strictly positive payoffs when an agreement occurs. Therefore, one can focus on the case where $g(P^*) > 0$, $g'(P^*) > 0$ and $g''(P^*) = 0$. In particular, given the restriction on $g(P^*)$ and $g'(P^*)$, the *foc* can be easily rearranged as $g'(P^*)P^*/g(P^*) = \lambda$ where the left hand side is positive. Therefore, one concludes that λ must be greater than 0 in order to have a maximum whereby

$$\lambda = \left(-(\mu-\sigma^2)+\sqrt{(\mu-\sigma^2)^2+2\sigma^2 r)}\right)/\sigma^2.$$

As already argued, this implies that for $t \leq \tau$, the state variable, P_t, will approach the trigger P^* from below, i.e. $P_t \leq P^*$.

Now, by these results, problems 24-25 become

$$E_t\alpha(P_{\tau_\alpha})e^{-(\tau_\alpha-t)\gamma} = \max_{P^*}\alpha(P^*)\left(\frac{P_t}{P^*}\right)^{\lambda_M} \tag{26}$$

and

$$E_t\beta(P_{\tau_\beta})e^{-(\tau_\beta-t)\delta} = \max_{P^*}\beta(P^*)\left(\frac{P_t}{P^*}\right)^{\lambda_U}, \tag{27}$$

[10]In the remainder of this section, we will discuss whether ξ should be taken positive or negative.

where

$$\lambda_M = \left(-(\mu - \sigma^2) + \sqrt{(\mu - \sigma^2)^2 + 2\sigma^2\gamma}\right)/\sigma^2$$

and

$$\lambda_U = \left(-(\mu - \sigma^2) + \sqrt{(\mu - \sigma^2)^2 + 2\sigma^2\delta}\right)/\sigma^2.$$

Solving the maximisation problems yields

$$P_M = arg\max_{P^*}\left\{\alpha(P^*)\left(\frac{P_t}{P^*}\right)^{\lambda_M}\right\} = \frac{\lambda_M}{\lambda_M - 1}\frac{(\gamma - \mu)(\delta + \gamma - 2\mu)\delta K}{(\delta - \mu)(\delta + \gamma)}, \tag{28}$$

$$P_U = arg\max_{P^*}\left\{\beta(P^*)\left(\frac{P_t}{P^*}\right)^{\lambda_U}\right\} = \frac{\lambda_U}{\lambda_U - 1}\frac{(\delta + \gamma - 2\mu)\gamma K}{(\delta + \gamma)}. \tag{29}$$

Observe that being γ and δ greater than μ, then λ_M and λ_U are greater than 1 and, hence, P_M and P_U are strictly positive.

It can be easily shown[11] that if $\gamma < \delta$, then $P_M > P_U$. This implies $\max\{\tau_\alpha, \tau_\beta\} = \tau_\alpha$. If instead $\gamma \geq \delta$ then $P_M \leq P_U$ and $\max\{\tau_\alpha, \tau_\beta\} = \tau_\beta$, then one concludes that at any initial level of the state variable $P_t \geq \max\{P_M, P_U\}$ there is immediate agreement yielding payoffs $(\alpha(P_t), \beta(P_t))$.

4.1 Timing and Efficiency of an Agreement

The aim of this subsection is to show that when P_t starts below $\max\{P_M, P_U\}$ an immediate agreement occurs at any initial level of the state variable satisfying $P_t > \underline{P}$ where, as we will show, $\underline{P} \leq \max\{P_M, P_U\}$. In other words, if P_t starts below $\underline{P}$ no immediate agreement is feasible and necessarily there will be a delay until P_t approaches $\underline{P}$. The question is whether or not this delay reflects Pareto inefficiency in the bargaining.

We will proceed as follows. By using the previous result, we will calculate $\underline{P}$, the "lower bound" to the initial level of the cash flow P_t, preventing delay. In doing this, we will show that Pareto optimality arises implicitly. This result implies that delay in agreement/investment is due to hysteresis[12] rather than any inefficiency in the bargaining process.

With a slight change of notation, we will first draw a general formulation and, at the end of this section, the specific results will be provided.

As proved in the previous section, for $P_t \geq \max\{P_M, P_U\}$ there is immediate agreement yielding payoffs $(\alpha(P_t), \beta(P_t))$; moreover $\max\{P_M, P_U\} = P_j$ where $j = M$ for $\gamma < \delta$ and

[11] See Cripps, 1997, p.544.

[12] See Dixit (1989).

$j = U$ for $\gamma \geq \delta$. Let player i be $\neq j$ and r_i and r_j be the time preferences (δ, γ). Also let $g(.) = \alpha(.)$ and $\underline{g}(.) = \beta(.)\delta/\gamma$ when $j = M$. Otherwise, let $g(.) = \beta(.)\delta/\gamma$ and $\underline{g}(.) = \alpha(.)$ when $j = U$[13]. If the initial level of the state variable, P_t, starts below P_j, then player j has an expected discounted SSP payoff equal to $g(P_j)(P_t/P_j)^{\lambda_j}$. Therefore, any earlier offer from player i, which does not guarantee $g(P_j)(P_t/P_j)^{\lambda_j}$ to j, will be rejected. On the other hand, if player i benefits from an earlier agreement, he will not offer more than j's reservation payoff. Then, if there is any advantage for i to make an early offer, this must yield $g(P_j)(P_t/P_j)^{\lambda_j}$ to j, and the residual payoff $P_t/(\gamma - \mu) - K - g(P_j)(P_t/P_j)^{\lambda_j}$ to i. Therefore, *if* there is any gain from an earlier agreement i makes an offer as soon as $P_t \geq \underline{P}$, where

$$\underline{P} = arg \max_P \left[\frac{P}{\gamma - \mu} - K - g(P_j)(P/P_j)^{\lambda_j} \right] (P_t/P)^{\lambda_i}. \tag{30}$$

Notice that, by using the stopping-problem technique introduced before, the threshold level $\underline{P}$ is the solution to the stopping problem

$$\sup_\tau E_t \left[\frac{P_\tau}{\gamma - \mu} - K - g(P_j)(P_\tau/P_j)^{\lambda_j} \right] e^{-r_i(\tau - t)}.$$

Moreover, there is a gain to i from making this offer iff

$$\begin{aligned} \max \left\{ \left[\frac{\underline{P}}{\gamma - \mu} - K - g(P_j)(\underline{P}/P_j)^{\lambda_j} \right] (P_t/\underline{P})^{\lambda_i}, \underline{g}(P_j)(P_t/P_j)^{\lambda_i} \right\} \\ = \left[\frac{\underline{P}}{\gamma - \mu} - K - g(P_j)(\underline{P}/P_j)^{\lambda_j} \right] (P_t/\underline{P})^{\lambda_i}. \end{aligned} \tag{31}$$

We will first show that 31 holds; this implies that if P_t starts sufficiently low there is an advantage in reaching an earlier agreement and not delaying until the cash flow crosses the threshold level P_j. Then we will show that condition 31 implies Pareto optimality.

Let $\bar{g}(P, P_t) = \left[\frac{P}{\gamma - \mu} - K - g(P_j)(P/P_j)^{\lambda_j} \right] (P_t/P)^{\lambda_i}$, by 30 it follows that $\bar{g}(\underline{P}, P_t) \geq \bar{g}(P', P_t)$ for any $P' \neq \underline{P}$ and any $P_t \leq \underline{P}$. Take $P' = P_j$, then

$$\begin{aligned} \bar{g}(\underline{P}, P_t) &\geq \bar{g}(P_j, P_t) \\ &= E_t \bar{g}(P_j, P_\tau = P_j) e^{-r_i(\tau - t)} \\ &= E_t \left[\frac{P_j}{\gamma - \mu} - K - g(P_j) \right] e^{-r_i(\tau - t)} \\ &= E_t \underline{g}(P_j) e^{-r_i(\tau - t)} \\ &= \underline{g}(P_j)(P_t/P_j)^{\lambda_i}. \end{aligned} \tag{32}$$

[13]Notice that these payoffs are defined as evaluated by the manager, this permits us to generalise the notation.

It is straightforward to notice that the equality holds, i.e. $\bar{g}(\underline{P}, P_t) = \underline{g}(P_j)(P_t/P_j)^{\lambda_i}$ iff $\gamma = \delta$, and in such a case $P_j = \max\{P_M, P_U\} = \underline{P}$ and there can be no earlier agreement for $P_t < P_j$.

Therefore, for any initial level of cash flows, P_t, satisfying $\underline{P} \leq P_t \leq P_j$ (with $\underline{P}$ defined by 30), there is an immediate agreement, which yields SSP outcomes to j and i (as evaluated by the manager) respectively equal to

$$g(P_j)(P_t/P_j)^{\lambda_j}, \tag{33}$$

$$\frac{P_t}{\gamma - \mu} - K - g(P_j)(P_t/P_j)^{\lambda_j}. \tag{34}$$

Implicitly, in this last calculation there arises the condition for the Pareto optimality of an earlier agreement yielding a SSP partition of the pie. In fact, one can use the following definition, with g and g' restricted to be SSP outcomes, and then compare the following definition to 32.. Pareto optimality[14]. Let $(g,\, \tau)$ be an outcome, where g is a share of the pie and τ is the random time at which agreement occurs. An outcome $(g,\, \tau)$ is Pareto optimal if there is no other outcome $(g',\, \tau')$ such that $E_t g' e^{-r\tau'} \geq E_t g e^{-r\tau}$ for any P_t with the strict inequality for some P_t.

Summary of SSP outcomes

We conclude this section summarising the SSP outcomes for any initial level of the state variable and values of γ and δ. Let S_M and S_U be the SSP payoffs to M and U respectively (as evaluated by the manager). Then we have the following results:

if $\mathbf{P_t \geq \max\{P_M, P_U\}}$

$$\alpha(P_t) = \frac{P_t}{\gamma - \mu}\frac{\delta - \mu}{\delta + \gamma - 2\mu} - K\frac{\delta}{\delta + \gamma}, \qquad \beta(P_t)\frac{\delta}{\gamma} = \frac{P_t}{\delta + \gamma - 2\mu} - K\frac{\gamma}{\delta + \gamma} \tag{35}$$

if $\mathbf{\underline{P} \leq P_t < \max\{P_M, P_U\}}$

* for $P_M > P_U$

$$S_M(P_t) = \alpha(P_M)(P_t/P_M)^{\lambda_M}, \qquad S_U(P_t) = \frac{P_t}{\gamma - \mu} - K - \alpha(P_M)(P_t/P_M)^{\lambda_M} \tag{36}$$

* for $P_U > P_M$

$$S_M(P_t) = \frac{P_t}{\gamma - \mu} - K - \beta(P_U)\frac{\delta}{\gamma}(P_t/P_U)^{\lambda_U}, \qquad S_U(P_t) = \beta(P_U)\frac{\delta}{\gamma}(P_t/P_U)^{\lambda_U} \tag{37}$$

[14] We use an ex ante definition of Pareto optimality as in Merlo and Wilson (1995, p.384).

if $\mathbf{P_t} < \underline{\mathbf{P}}$ there is no immediate agreement and the expected discounted payoffs of an agreement at $\underline{P}$ are:

* for $P_M > P_U$

$$S_M(P_t) = \alpha(P_M)(P_t/P_M)^{\lambda_M}, \qquad S_U(P_t) = \left[\frac{\underline{P}}{\gamma-\mu} - K - \alpha(P_M)(\underline{P}/P_M)^{\lambda_M}\right](P_t/\underline{P})^{\lambda_U} \tag{38}$$

* for $P_U > P_M$

$$S_M(P_t) = \left[\frac{\underline{P}}{\gamma-\mu} - K - \beta(P_U)\frac{\delta}{\gamma}(\underline{P}/P_U)^{\lambda_U}\right](P_t/\underline{P})^{\lambda_M}, \qquad S_U(P_t) = \beta(P_U)\frac{\delta}{\gamma}(P_t/P_U)^{\lambda_U}. \tag{39}$$

In the above results P_M and P_U are given by

$$P_M = \frac{\lambda_M}{\lambda_M - 1}\frac{(\gamma-\mu)(\delta+\gamma-2\mu)\delta K}{(\delta-\mu)(\delta+\gamma)}, \qquad P_U = \frac{\lambda_U}{\lambda_U - 1}\frac{(\delta+\gamma-2\mu)\gamma K}{(\delta+\gamma)},$$

with $P_M > P_U$ iff $\delta > \gamma$.

Moreover, one may notice that, for $\delta > \gamma$, $\underline{P}$ is the solution to

$$S_M(\underline{P}) = \frac{\underline{P}}{\gamma-\mu}\frac{\lambda_U - 1}{\lambda_U - \lambda_M} - K\frac{\lambda_U}{\lambda_U - \lambda_M}, \tag{40}$$

and for $\delta < \gamma$ the solution to

$$S_U(\underline{P}) = \frac{\underline{P}}{\gamma-\mu}\frac{\lambda_M - 1}{\lambda_M - \lambda_U} - K\frac{\lambda_M}{\lambda_M - \lambda_U}, \tag{41}$$

where 40 and 41 can be derived by rearranging the first order conditions of problem 30. Last, for $\delta = \gamma$

$$\underline{P} = P_M = P_U. \tag{42}$$

In Figure 1, we show the equilibrium outcome of the sequential bargaining when $\delta > \gamma$ and, hence, $P_M > P_U$. In this case, it is intuitive that the more patient player, – the manager –, should guarantee a larger expected payoff. In fact, as expected, the blue line, representing $S_M(P_t)$, is always above the red line, which represents $S_U(P_t)$. Last, notice that according to 3.41, $S_U(P_t)$ 'smooth-pastes' the grey, continuous curve, representing the residual payoff $\frac{P_t}{\gamma-\mu} - K - S_M(P_t)$, and the tangency between the two curves is guaranteed by the fact that $\underline{P}$ maximises the union's expected payoff.

5 Comparison with the Nash Bargaining Solution

We are now ready to give a technical result, but nonetheless useful when one wants to compare the SSP outcome of sequential bargaining with the Nash bargaining outcome.

A more simple bargaining procedure was introduced by Nash (1950) and the bargaining protocol is described as follows. Each player simultaneously demands a share of the pie; if the demands are feasible, given the size of the pie, then the players receive their demands otherwise they receive their disagreement payoffs. The main difference with respect to the sequential bargaining is that the Nash protocol implies the existence of a cooperative framework where players can commit to demands and disagreement payoffs. Most importantly, one can imagine an umpire who guarantees players will stick to their commitments. Then the cooperative framework is meant to enforce threats and commitments[15].

As the Nash bargaining outcome is not the central issue of this paper, we will only introduce the main conditions which permit one to derive the Nash bargaining solution, and we refer the reader to Nash (1950, 1953).

Nash (1953) stated axioms which the solution to this bargaining situation should satisfy, and, in turn, these axioms allow one to restrict the set of solutions to a single point. It can be proved that the Nash bargaining solution is the only solution to the above bargaining situation, which satisfies these axioms. In particular, the Nash bargaining solution can be derived by maximising the product of each agent's surplus[16].

We will define the Nash bargaining in a stochastic environment following the characterisation by Perraudin and Psillaki (1999).

In our bargaining situation, each agent's surplus corresponds to a demand, say, D_M, D_U and the disagreement payoffs is zero when no wage-contract has been agreed upon. In particular, the expected discounted payoffs of an agreement are $D_M(P_t) = [P/(\gamma-\mu) - K - w/\gamma](P_t/P)^{\lambda_M}$ and $D_U(P_t) = w/\delta(P_t/P/)^{\lambda_U}$ and the Nash problem can be formalised as

$$D^*_M(P_t)D^*_U(P_t) = \max_{P,w}\left[\frac{P}{\gamma-\mu} - K - \frac{w}{\gamma}\right]\left(\frac{P_t}{P}\right)^{\lambda_M}\frac{w}{\delta}\left(\frac{P_t}{P}\right)^{\lambda_U}. \tag{43}$$

[15]See van Damme (1991, p.160) for a discussion on commitment and cooperation regarding the Nash bargaining solution.

[16]See Binmore and Dasgupta (1987, pp.32-37)

By the first order conditions one obtains

$$\frac{\partial D_M/\partial P}{\partial D_M/\partial w} = \frac{\partial D_U/\partial P}{\partial D_U/\partial w}, \tag{44}$$

$$\frac{\partial D_M/\partial w}{D_M} = -\frac{\partial D_U/\partial w}{D_U}. \tag{45}$$

Observe that 44 guarantees Pareto optimality of the outcome[17] in that each agent has the same trade off (marginal rate of substitution) between the timing of the agreement and payoff from the agreement. Then one can refer to 44 as Pareto optimality (or efficiency) condition.

Condition 45 is related to the bargaining power of players in the negotiation. If any asymmetry in the negotiation is correctly reflected in the agreement payoffs and the bargaining protocol is symmetric[18], then agents have the same bargaining power[19]. We will refer to 45 as Symmetry condition. Perraudin and Psillaki (1999) use the generalised version of the Nash bargaining with an exogenous parameter capturing any asymmetry in the bargaining protocol so they refer to this condition as Bargaining condition.

From 44-45, one can derive the Nash bargaining outcome (P^* and w^*), which allows us to make an immediate comparison with the sequential bargaining[20].

From the Symmetry condition, players split the pie a half and a half, then

$$D_M(P^*) = D_U(P^*)\delta/\gamma = \left(\frac{P^*}{\gamma-\mu} - K\right)/2 \tag{46}$$

and the Pareto optimality condition yields a share

$$D_U(P^*)\frac{\delta}{\gamma} = \frac{P^*}{\gamma-\mu}\frac{\lambda_M - 1}{\lambda_M - \lambda_U} - K\frac{\lambda_M}{\lambda_M - \lambda_U} \tag{47}$$

to U, and a share

$$D_M(P^*) = \frac{P^*}{\gamma-\mu}\frac{\lambda_U - 1}{\lambda_U - \lambda_M} - K\frac{\lambda_U}{\lambda_U - \lambda_M} \tag{48}$$

to M.

The comparison with the sequential bargaining is quite immediate. Just notice that the Rubinstein SSP outcome is Pareto optimal at $P_t = \underline{P}$ and therefore it must satisfy

[17]Pareto efficiency is one of axioms which the Nash bargaining solution satisfies.

[18]In the sense that the bargaining procedure -for instance the sequence of moves- does not give any advantage to either player.

[19]See Nash (1953, p.138).

[20]See also Binmore, Rubinstein and Wolinsky (1986) for a comparison between the Rubinstein and the Nash bargaining in the deterministic case.

the efficiency conditions 47-48. In fact, as already found in 40 and 41, at $P_t = \underline{P}$, the SSP equilibrium shares (for any $\delta \neq \gamma$) are

$$S_M(\underline{P}) = \frac{\underline{P}}{\gamma - \mu} \frac{\lambda_U - 1}{\lambda_U - \lambda_M} - K \frac{\lambda_U}{\lambda_U - \lambda_M}, \tag{49}$$

and

$$S_U(\underline{P}) = \frac{\underline{P}}{\gamma - \mu} \frac{\lambda_M - 1}{\lambda_M - \lambda_U} - K \frac{\lambda_M}{\lambda_M - \lambda_U}, \tag{50}$$

to M and U respectively, and these are nothing but the efficiency conditions of the Nash bargaining problem (see 47, 48). Geometrically, this implies that there exists a straight line, characterising a Pareto efficient outcome and represented by the function

$$\frac{P_t}{\gamma - \mu} \frac{\lambda_i - 1}{\lambda_i - \lambda_j} - K \frac{\lambda_i}{\lambda_i - \lambda_j},$$

which crosses the efficient outcome $(D_i(P^*),\ P^*)$ of the Nash bargaining and the efficient outcome $(S_i(\underline{P}),\ \underline{P})$ of the sequential bargaining for player $i = M, U$ and $j \neq i$. This comparison is depicted in Figure 2, where, in particular, we show the case where $\delta > \gamma$, that is the labour union is more impatient than the manager.

Moreover, as shown in Figure 15, one can notice that the slope of the line $\frac{P_t}{\gamma-\mu} \frac{\lambda_U - 1}{\lambda_U - \lambda_M} - K \frac{\lambda_U}{\lambda_U - \lambda_M}$ is positive when $\delta > \gamma$ which, in turn, implies $\lambda_U > \lambda_M$ (and λ_i is always greater than 1). Combining this with the fact that in a sequential bargaining the more patient player can always guarantee a bigger share of the pie, implies that the efficient outcome in the Nash bargaining occurs at a lower level of cash flow than the efficient outcome in the sequential bargaining. A similar reasoning applies in the reverse case where $\delta < \gamma$ and the same conclusion holds, that is Nash bargaining accelerates agreement. In other words, this means that $P^* < \underline{P}$ for $\gamma \neq \delta$ (and $P^* = \underline{P} = P_M = P_U$ for $\delta = \gamma$) and, hence, the lack of a cooperative environment delays agreements and investment decisions.

Conclusions

The main result provided by this paper refers to the uniqueness of the subgame perfect equilibrium in a sequential bargaining where the underlying surplus follows a geometric Brownian motion. The uniqueness of the equilibrium implies that at any initial level of the state variable, players always play the same SSP strategy. The only requirement in order to obtain this result is the restriction over the time preferences, bounded to be greater than the drift of the stochastic process. The SSP equilibrium outcome found does not differ from that derived by Cripps. However, the central issue to the present paper consists of proving the uniqueness of such an outcome.

The additional contribution of the current paper is to highlight other features of the equilibrium. If the initial level of the state variable is sufficiently low players wait until the cash flows increase up to a trigger level $\underline{P}$, where agreement occurs. This agreement partition is efficient as it satisfies the Pareto optimality condition. Therefore a delay in the agreement is due to the fact that not any state contingent outcome can be agreed upon in the current period. This means that for a low level of cash flows, the sum of the expected agreement payoffs may exceeds the overall value of the agreement surplus (which can be negative because of an initial fixed cost). In fact, it is intuitive that if the fixed cost, K, is zero, there is an immediate agreement for any initial level of the cash flow (just notice that $P_j = \max\{P_M, P_U\}$ tends to zero as $K = 0$).

Furthermore, a technical, but nonetheless useful, result is achieved by providing a comparison between the sequential and the Nash bargaining outcome. This comparison can contribute to explicitly address timing considerations. In fact, the unique equilibrium entry trigger derived under the non-cooperative scenario cannot be solved analytically. However, as the Nash and the Rubinstein outcomes are Pareto efficient, a relation between the two investment threshold levels can be established. This relation highlights the investment delay arising in a non-cooperative bargaining compared to the cooperative one. The lack of an exogenous mechanism aimed at enforcing contracts results into entry delay rather than into inefficiency.

Last, we stress that in the current paper we have analysed the relation between the Rubinstein and the Nash *symmetric* bargaining with the purpose of highlighting timing considerations of the Rubinstein outcome. However, in the deterministic case, the relation between the sequential bargaining and the *asymmetric* Nash bargaining has been the object of great effort in game theory. The asymmetric (or generalised) Nash bargaining takes into account any asymmetry by means of an exogenous parameter representing the relative bargaining power. As the time interval between offers and counteroffers tends to zero, a well known result to game theory scholars is that the Rubinstein outcome approaches the Nash solution with bargaining power equal to the relative time preferences of the Rubinstein model. Whether uncertainty may affect this result is left for further research.

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FIGURE 1: Alternating-offer equilibrium values when $\delta > \gamma$

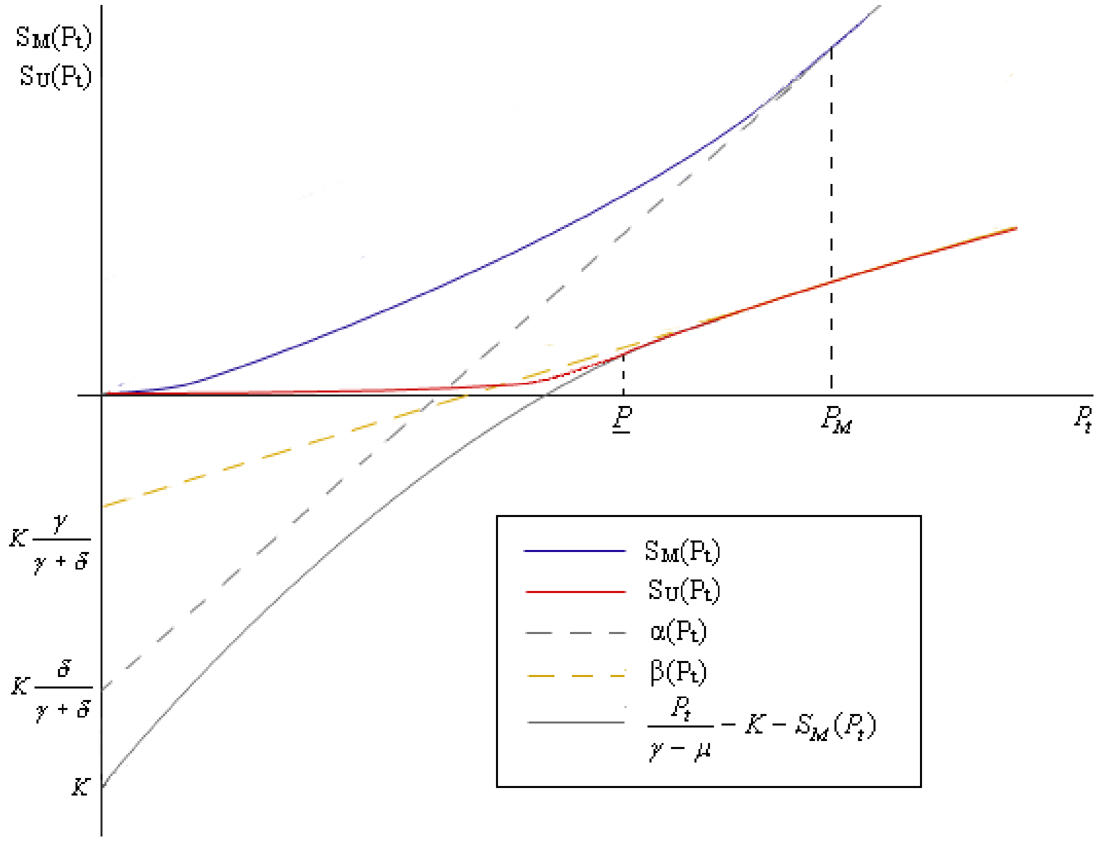

FIGURE 2: Alternating-offer equilibrium values and Nash Bargaining solution when $\delta > \gamma$