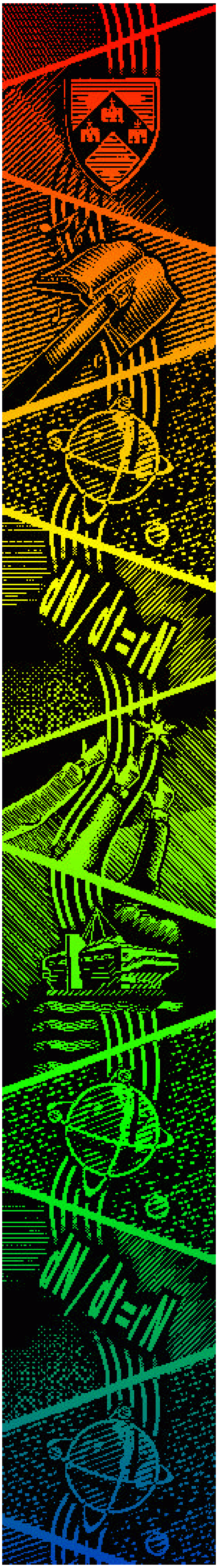

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by

Menelaos Karanasos

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

THE COVARIANCE STRUCTURE OF THE S-GARCH AND M-GARCH MODELS

Menelaos Karanasos ⋆

University of York, Heslington, York, YO10 5DD, UK

Abstract

The purpose of this paper is to examine the variance-covariance structure of multivariate GARCH models that have been introduced in the literature the last decade, and have been greatly favoured by time series analysts and econometricians. In particular, we analyze the second moments of the sum of GARCH models (S-GARCH) examined in Karanasos, Psaradakis and Sola (1999), and the multivariate GARCH models (M-GARCH) introduced by Bollerslev, Engle and Wooldridge (1988), Bollerslev (1990) and Engle and Kroner (1995).

Key Words: Autocovariances, Multivariate GARCH, Autocovariance Generating Function, Wold Representation.

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⋆ For correspondence: Email: mk16@york.ac.uk, Tel: 01904 433799, Fax: 01904 433759.

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1 Introduction

One of the most popular class of non-linear processes are the generalized autoregressive conditional heteroscedasticity models[1] (GARCH). The existence of the huge literature which uses these processes in modelling conditional volatility in high frequency financial assets demonstrates the popularity of the various GARCH models (see, for example, the recent surveys by Bollerslev, Chou and Kroner (1992), Bera and Higgins (1993), Bollerslev, Engle and Nelson (1994), Diebold and Lopez (1995), Engle (1995), Palm (1996), Shephard (1996) and Pagan (1996); see also the book by Gourieroux (1997) for a detail discussion of the GARCH models and financial applications). Parallel to the success of the standard linear ARMA-type of time series models arising from the use of the conditional versus the unconditional mean, the key insight offered by the GARCH model and its various extensions, lies in the distinction between the conditional and the unconditional second order moments.

Since leptokurtic distributions of asset returns and slowly decaying autocovariances of squared returns are among the stylized facts of financial high-frequency time-series, theoretical expressions for the unconditional second moments of GARCH models are of both statistical interest and of practical significance. Specifically, given the expressions for the second moments of the squared errors, practitioners can compare the estimates of the kurtosis of the errors and the autocovariances of the squared errors obtained directly from the data with those obtained from their model. This enable them to decide on how well their GARCH model fits the data. In addition, given the results on the existence of the second moments, the practitioner can check what his/her estimates imply about those moments. The fourth moment of the errors and the covariance structure of the squared errors and of the conditional variance of the GARCH(p,q) model have already been given in the literature (see Karanasos, 1999 a,e, hereafter K, and He and Terasvirta, 1997, hereafter HT.).

As economic variables are inter-related, generalisation of univariate models to the multivariate set-up is quite natural-this is more so for the GARCH[2] models. The motivation for the multivariate GARCH models (MGARCH) stems from the fact that many economic variables react to the same information, and hence, have nonzero conditional covariances. Thus, multivariate GARCH-type models provide a natural framework for analyzing the joint dynamic behaviour of volatility of financial assets and/or markets and they are particular useful in multivariate financial models (such as the CAPM or dynamic hedging models). In the MGARCH models the conditional covariance matrix (ccm) depends non-trivially on the past of the process[3].

From the many different multivariate functional forms the diagonal GARCH model

[1]The ARCH model was originally proposed by Engle (1982), whereas Taylor (1986) and Bollerslev (1986), hereafter B, independently of each other, presented the generalised ARCH model.

[2]The first paper on multivariate GARCH models was written by Engle, Granger and Kraft (1984). They used a bivariate ARCH(1) process to combine forecasts in two models of US inflation.

[3]The MGARCH model specifies the conditional variances/covariances as a linear function of squared innovations and past conditional variances/covariances. Although the processes themselves are nonlinear, the conditional variances and covariances are linear. This modeling strategy leads to two major problems: on the one hand we must establish sufficient conditions to ensure the positive definiteness (pd) of the ccm, while on the other, to make the estimation of the model feasible we must find a parsimonious representation of the data with a reasonable number of parameters to be estimated. For these reasons various alternative representations have been proposed in the literature.

originally suggested by Bollerslev, Engle and Wooldridge (1988), hereafter BEW, and the constant conditional correlation (ccc) GARCH put forward in B (1990), have become perhaps the most common. BEW specified that the conditional variances/covariances depend only on its own lagged errors and lagged values[4]. In B's representation the ccm is time varying, but the conditional correlations are assumed to be constant[5].

The preceding models appear often in the literature of the late eighties early nineties though usually without any theoretical discussion. In particular, various cases of the diagonal representation of the MGARCH model (with various mean specifications) have been applied by many researchers. For example, it has been used by BEW(1988) for their analysis of returns on bills, bonds and stocks, by Engel and Rodrigues (1989) to test the international CAPM, by Kaminsky and Peruga (1990) to examine the risk premium in the forward market for foreign exchange, by McCurby and Morgan (1991) to test the uncovered interest rate parity, and by Baillie and Myers (1991) to estimate optimal hedge ratios in commodity markets. B (1990) illustrated the validity of the ccc MGARCH model for a set of five nominal European U.S. dollar exchange rates following the inception of the European Monetary System. The ccc MGARCH model has also been used by Cecchetti, Cumby and Figlewski (1988) to estimate the optimal future hedge, McCurby and Morgan (1989) to examine risk premia in foreign currency futures market, Schwert and Seguin (1990) to analyze stock returns, Baillie and Bollerslev (1990) to model risk premia in forward foreign exchange rate markets, Kroner and Claessens (1991) to analyze the optimal currency composition of external debt, Ng (1991) to test the CAPM, and Kroner and Sultan (1993) to estimate futures hedge ratios.

Although the MGARCH models were introduced almost a decade ago and have been widely used in empirical applications , its statistical properties have only recently been examined by researchers. EK(1995) examined the identification and maximum likelihood estimation of the vec, diagonal and BEKK representations of the MGARCH model. Lin (1997), hereafter L, provided a comprehensive analytical tool for the impulse response analysis for all the aforementioned representations of the MGARCH model. Tse (1998) developed the Lagrange multiplier test for the hypothesis of constant correlation in Bollerslev's representation whereas Jeantheau (1998), and Ling and McAleer (1999) investigated the asymptotic theory of the quasi maximum likelihood estimator for an extension of the ccc MGARCH model.

However, the analysis of the covariance structure of the MGARCH model has not been considered yet. This article attempts to fill in this gap in the GARCH literature. The focus will be on the fourth moment of the errors and on the theoretical acf of the squared errors and of the conditional variances/covariances. In this context, the paper generalizes the results for the univariate GARCH model given in K (1999 a,e) and HT(1997) to various multivariate GARCH models.

In Section 2 we present the covariance structure of the conditional variances/covariances

[4]These restrictions are intuitively reasonable and reduce the large number of parameters in the vec MGARCH model (introduced by Engle and Kroner (1995), hereafter EK). Moreover, in the vec representation, and even in the diagonal representation, the conditions to ensure that the ccm are positive definite (pd) a.s. for all t are difficult to impose and verify. Therefore, EK (1995) proposed a new parametrisation (they refer to it as the BEKK representation) that easily imposes these restrictions.

[5]This assumption greatly simplifies the computational burden in estimation, and conditions for the ccm to be pd a.s. for all t are also easy to impose.

and of the squared errors for the S-GARCH[6] process (this is the sum of n processes which follow a diagonal MGARCH model). Our results include as a special case (when $n = 1$) the covariance structure of the conditional variance and of the squared errors for the GARCH(p,q) model presented in K (1999a), HT(1997) and K(1999e). In Section 3 we contribute to the theoretical developments in the multivariate GARCH literature by presenting the theoretical acf of the conditional variances/covariances and of the squared errors of the vec[7] and the ccc representations of the MGARCH models.

The goal of this article is to provide a comprehensive methodology for the analysis of the covariance structure in multivariate GARCH models. First, it derives the VARMA representations of the conditional variances/covariances and of the squared errors and it gives general conditions for stationarity, invertibility and identifiability of these representations. Second, it provides the univariate ARMA representations of the conditional variances/covariances[8] and of the squared errors. Third, it uses two alternative (and equivalent) methods to obtain the autocovariances: (i) the Wold representation (wr) of a stationary stochastic process (ssp) and (ii) the canonical factorization (cf) of the autocovariance generating function (agf) of a ssp[9]. It should be noted that we only examine the case of distinct roots in the AR polynomials of the univariate ARMA representations and we express the autocovariances in terms of the roots of the AR polynomials and the parameters of the MA polynomials of the univariate ARMA representations.

[6]The S-GARCH model was introduced by Karanasos, Psaradakis and Sola (1999), hereafter KPS, who applied it to option pricing.

[7]The vec representation includes the diagonal and the BEKK representations as special cases.

[8]The proof is not presented in this paper. For an analytical derivation of the univariate ARMA representations see K(1999c).

[9]To our knowledge this is the first paper that applies the wr and the agf of a stochastic process to a multivariate GARCH model. These have been widely used in the ARMA-VARMA literature (see, for example, Pandit (1973, p. 99), Nerlove, Grether and Carvalho (1979, pp. 30-43, 39, 70-85), Pandit and Wu (1983, pp. 87-89, 105, 129-130), Brockwell and Davis (1987, pp. 87-89, 102-103, 180-182, 408-410), Reinsel (1993, pp. 7, 33-34), Hamilton(1994, pp. 59-60, 61-63) and K(1999b)).

2 SUM OF GARCH MODELS

Let y_t be equal to the sum of n processes

$$y_t = \sum_{i=1}^{n} y_{it}, \;\; y_{it} = \mu_{i,t-1} + \epsilon_{it} \tag{2.1}$$

The ϵ_{it} 's follow a diagonal multivariate GARCH process:

$$\bar{\epsilon}_{t|t-1} \sim N(0, \bar{h}_t), \;\; \bar{\epsilon}_t = \begin{bmatrix} \epsilon_{1t} \\ . \\ . \\ . \\ \epsilon_{nt} \end{bmatrix}, \;\; \bar{h}_t = \begin{bmatrix} h_{1t} & h_{12,t} & h_{1n,t} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ h_{n1,t} & \cdots & h_{nt} \end{bmatrix} \tag{2.2}$$

Each h_{it} follows a GARCH(p_i, q_i) process

$$B_i(L)h_{it} = \omega_{i0} + A_i(L)\epsilon_{it}^2, \;\; \text{where} \tag{2.3}$$

$$B_i(L) = -\sum_{k=0}^{p_i} \beta_{ik} L^k, \;\; A_i(L) = \sum_{k=1}^{q_i} a_{ik} L^k, \;\; \beta_{i0} = -1 \tag{2.3a}$$

For simplicity and without loss of generality we will assume that the conditional covariances are constant: $h_{mk,t} = \omega_{mk,0}, \; m, k = 1 \cdots, n$

Corollary 1a. The ARMA representations of the conditional variances (h_{it}) *are given by*

$$B_i^\star(L)h_{it} = \omega_{i0} + A_i(L)v_{it}, \;\; v_{it} = \epsilon_{it}^2 - h_{it} \tag{2.4}$$

$$B_i^\star(L) = B_i(L) - A_i(L) = \prod_{j=1}^{p_i^\star}(1 - \beta'_{ij}L), \;\; p_i^\star = max(p_i, q_i) \tag{2.4a}$$

The v_{it}'s are uncorrelated (although not independent) terms. The covariance matrix of the v_{it}'s is denoted by [10]

$$\bar{\sigma}_v = \begin{bmatrix} \sigma_{11,v} & \sigma_{12,v} & \sigma_{1n,v} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \sigma_{n1,v} & \cdots & \sigma_{nn,v} \end{bmatrix} \tag{2.5}$$

[10] The derivation of the covariances of the v_{it}'s is presented in the next Section.

Proof. In (2.3) we add and subtract $A_i(L)h_{it}$ and we get (2.4).

Assumption 1. All the roots of the autoregressive polynomials $(B_i^\star(L))$ *and all the roots of the moving average polynomials* $(A_i(L))$ *are lie outside the unit circle (Stationarity and Invertibility conditions).*

Assumption 2. The polynomials $B_i^\star(L)$ *and* $A_j(L)$ $(i,j=1,\cdots,n)$ *are left coprime. In other words the representation* $\frac{B_i^\star(L)}{A_j(L)}$ *is irreducible.*

Corollary 1b. The conditional variance of y_t *is equal to the sum of the* n *GARCH processes* h_{it}[11]*:*

$$h_t = V_{t-1}(y_t) = \omega_0^\star + \sum_{i=1}^{n} h_{it} \tag{2.6}$$

The $\omega_0^\star$ together with the proof is given in Appendix A.

Theorem 1. Under assumptions 1 and 2, the autocovariance generating function (agf) of the preceding process (h_t) *is given by* [12]*:*

$$g_z(h) = \sum_{m=0}^{\infty} f_m \gamma_m (z^m + z^{-m}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{A_i(z)A_j(z^{-1})}{B_i^\star(z)B_j^\star(z^{-1})}\sigma_{ij,v}, \ \text{where} \tag{2.7}$$

$$\gamma_m = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(\sum_{l=1}^{p_i^\star} \zeta_{il,j}^m \lambda_{l,m}^{ij} + \sum_{k=1}^{p_j^\star} \zeta_{jk,i}^m \lambda_{ij}^{k,m}\right)\sigma_{ij,v}, \quad f_m = \begin{cases} .5 & \text{if } m=0 \\ 1 & \text{otherwise} \end{cases}, \tag{2.7a}$$

$$\zeta_{il,j}^m = \frac{\zeta_{il}^m}{\prod\limits_{k=1}^{p_j^\star}(1-\beta'_{il}\beta'_{jk})}, \quad \zeta_{il}^m = \frac{(\beta'_{il})^{p_i^\star - 1 + m}}{\prod\limits_{\substack{k=1\\k\neq l}}^{p_i^\star}(\beta'_{il}-\beta'_{ik})}, \tag{2.7b}$$

$$\lambda_{l,m}^{ij} = \sum_{c=0}^{q_j}\sum_{d=1}^{q'_i} a_{id}a_{j,d+c}(\beta'_{il})^c + \sum_{c=1}^{m^\star}\sum_{d=1}^{q'_j} a_{jd}a_{id+c}(\beta'_{il})^{-c}, \quad \lambda_{ij}^{k,m} = \sum_{c=m+1}^{q_i}\sum_{d=1}^{q'_j} a_{jd}a_{id+c}(\beta'_{jk})^{c-2m} \tag{2.7c}$$

and

$$\begin{bmatrix} p_i^\star \\ q'_i \\ q'_j \\ m^\star \end{bmatrix} = \begin{bmatrix} \max(p_i, q_i) \\ \min(q_i, q_j - c) \\ \min(q_j, q_i - c) \\ \min(m, q_i) \end{bmatrix} \tag{2.7d}$$

The proof of Theorem 1 is presented in Appendix A.

[11] To our knowledge KPS (1999) were the first to analyze sum of GARCH processes.

[12] To our knowledge this is the first paper that uses the cf of the agf to analyse the covariance structure of the MGARCH model. For an application of this method to the VARMA model see, for example, K (1999b).

3 MULTIVARIATE GARCH MODELS

3.1 Vec Representation

In what follows we examine the vec representation of the Multivariate GARCH(r,s) process of order p [MGARCH(r,s,p)].

The general form of the MGARCH(r,s,p) model is given by

$$\bar{h}_t^\star = \Omega + \sum_{l=1}^{s} {}_lA\, \vec{\epsilon}_{t-l}^{\,\star} + \sum_{l=1}^{r} {}_lB^\star\, \bar{h}_{t-l}^\star, \text{ where} \tag{3.1}$$

$$\bar{h}_t^\star = \begin{bmatrix} h_{1t}^\star \\ \cdot \\ \cdot \\ \cdot \\ h_{p^\star t}^\star \end{bmatrix}, \quad \vec{\epsilon}_{t-l}^{\,\star} = \begin{bmatrix} \epsilon_{1t-l}^\star \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_{p^\star t-l}^\star \end{bmatrix}, \quad \Omega = \begin{bmatrix} \omega_1^\star \\ \cdot \\ \cdot \\ \cdot \\ \omega_{p^\star} \end{bmatrix}, \quad p^\star = \frac{p(p+1)}{2} \tag{3.1a}$$

$${}_lA = \begin{bmatrix} a_{11}^l & \dots & a_{1p^\star}^l \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{p^\star 1}^l & \dots & a_{p^\star p^\star}^l \end{bmatrix}, \quad {}_lB^\star = \begin{bmatrix} b_{11}^{l\star} & \dots & b_{1p^\star}^{l\star} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ b_{p^\star 1}^{l\star} & \dots & b_{p^\star p^\star}^{l\star} \end{bmatrix} \tag{3.1b}$$

and

$$h_{it}^\star = \begin{cases} h_{it} & \text{if } i \le p, \\ h_{lk,t} & \text{if } i = l \cdot p + k - \frac{l(l+1)}{2}, \end{cases} \tag{3.1c}$$

$$\epsilon_{i,t-l}^\star = \begin{cases} \epsilon_{i,t-l}^2 & \text{if } i \le p, \\ \epsilon_{l,t-l}\epsilon_{k,t-l} & \text{if } i = n \cdot p + k - \frac{n(n+1)}{2} \end{cases} \tag{3.1d}$$

$$i = 1, \cdots, p^\star, \quad \begin{array}{rcl} n & = & 1, \cdots, p-1, \\ k & = & n+1, \cdots, p \end{array}$$

An alternative expression for the MGARCH(r,s,p) is:

$$B^{L\star}\bar{h}_t^\star = \Omega + A^L \vec{\epsilon}_t^{\,\star}, \text{ where} \tag{3.2}$$

$$B^{L\star} = \begin{bmatrix} 1 - B_{11}^{L\star} & \dots & -B_{1p^\star}^{L\star} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ -B_{p^\star 1}^{L\star} & \dots & 1 - B_{p^\star p^\star}^{L\star} \end{bmatrix}, \quad A^L = \begin{bmatrix} A_{11}^L & \dots & A_{1p^\star}^L \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{p^\star 1}^L & \dots & A_{p^\star p^\star}^L \end{bmatrix}, \text{ and} \tag{3.2a}$$

$$B_{ik}^{L\star} = \sum_{l=1}^{r} b_{ik}^{l\star} L^l, \quad A_{ij}^L = \sum_{l=1}^{s} a_{ij}^l L^l \tag{3.2b}$$

Corollary 2a. The VARMA representation of the conditional variances (and covariances) for the MGARCH(r,s,p) model is given by

$$\bar{h}_t^\star = \Omega + \sum_{l=1}^{r'} {}_lB\,\bar{h}_{t-l}^\star + \sum_{l=1}^{s} {}_lA\,\bar{v}_{t-l}^\star, \;\; r' = max(r,s) \tag{3.3}$$

where

$$ {}_lB = \begin{cases} {}_lB^\star + {}_lA & \text{if } l < r, s \\ {}_lB^\star & \text{if } l, r > s \\ {}_lA & \text{if } l, s > r \end{cases}, \;\; \bar{v}_{t-l}^\star = \begin{bmatrix} v_{1,t-l}^\star \\ . \\ . \\ . \\ v_{p^\star,t-l}^\star \end{bmatrix} \tag{3.3a}$$

$v_{i,t-l}^\star$ are uncorrelated (although not independent) terms. They are given by

$$v_{i,t-l}^\star = \begin{cases} \epsilon_{i,t-l}^2 - h_{i,t-l} & \text{if } \; i \le p \\ \epsilon_{n,t-l}\epsilon_{k,t-l} - h_{nk,t-l} & \text{if } \begin{cases} i = np + k - \frac{n(n+1)}{2} \\ n = 1, \cdots, p-1 \\ k = 1 \cdots, p \end{cases} \end{cases} \tag{3.3b}$$

Their covariance matrix is denoted by[13]

$$\bar{\sigma}_v^\star = \begin{bmatrix} \sigma_{11,v} & \sigma_{1p^\star,v} \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \sigma_{p^\star 1,v} & \sigma_{p^\star p^\star,v} \end{bmatrix} \tag{3.3c}$$

An alternative expression for the VARMA representation is given by:

$$B^L \bar{h}_t^\star = \Omega + A^L \bar{v}_t^\star, \;\; \text{where} \tag{3.4}$$

$$B^L = \begin{bmatrix} 1 - B_{11}^L & \ldots & -B_{1p^\star}^L \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -B_{p^\star 1}^L & \ldots & 1 - B_{p^\star p^\star} \end{bmatrix}, \;\; B_{ik}^L = \sum_{l=1}^{r'} b_{ik}^l L^l, \;\; b_{ik}^l = \begin{cases} b_{ik}^{l\star} + a_{ik}^l & \text{if } \; l < r, s \\ b_{ik}^{l\star} & \text{if } \; l, r > s \\ a_{lk}^l & \text{if } \; l, s > r \end{cases} \tag{3.4a}$$

Proof. In (3.1) we add and subtract $\sum_{l=1}^{s} {}_lB^\star\,\bar{h}_{t-l}^\star$ to get (3.3).

[13]The derivation of the $\sigma_{ij,v}$'s is given in Appendix B.

Corollary 2a′. The VARMA representation of the squared errors for the MGARCH(r,s,p) models is given by

$$\vec{\epsilon}^{\star}_t = \sum_{l=1}^{r'} {}_lB\,\vec{\epsilon}^{\star}_{t-l} + \sum_{l=0}^{r} {}_lB^{\star}\,\vec{v}^{\star}_{t-l}, \text{ or } B^L\vec{\epsilon}^{\star}_t = \Omega + B^{L\bullet}\vec{v}^{\star}_t, \text{ where} \tag{3.5}$$

$$B^{L\bullet} = \begin{bmatrix} B^{L\star}_{11} & \dots & B^{L\star}_{1p^\star} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ B_{p^\star 1} & \dots & B^{L\star}_{p^\star p^\star} \end{bmatrix} \tag{3.5a}$$

Proof. In (3.1) we add and subtract $\sum_{l=0}^{r} {}_lB^{\star}\,\vec{\epsilon}^{\star}_{t-l}$ (where ${}_0B^{\star} = -I$) to get (3.5).

Corollary 2b. The univariate ARMA representations of the conditional variances (and covariances) are given by

$$\prod_{l=1}^{\bar{p}}(1 - B_lL)h^{\star}_{it} = \sum_{j=1}^{p^\star}\sum_{l=1}^{\tilde{p}} B^{ij}_l L^l v^{\star}_{jt} \tag{3.6}$$

where $\bar{p} = p^\star \times r'$ and $\tilde{p} = (p^\star - 1)r' + s$. The B_l's and B^{ij}_l's are given in Appendix B (see eqs B.1-B.1b) and have been derived in K (1999c).

Corollary 2b′. The univariate ARMA representations of the squared errors are given by

$$\prod_{l=1}^{\bar{p}}(1 - B_lL)\epsilon^{\star}_{it} = \sum_{j=1}^{p^\star}\sum_{l=0}^{\hat{p}} A^{ij}_l L^l v^{\star}_{jt} \tag{3.7}$$

where $\hat{p} = (p^\star - 1)r' + r$. The B_l's are as in (3.6). The A^{ij}_l's can be derived from (3.5) in an analogous way to the B^{ij}_l's in (3.6).

Assumption 3. All roots of $|B^L|$ and all roots of $|A^L|$ lie outside the unit circle. These conditions satisfy the stationarity and invertibility of the model.

Assumption 4. The polynomials B^L and A^L have no common left factors other than unimodular ones (this condition satisfy the irreducibility of the model) and satisfy other identificiability conditions given in Dunsmuir and Hannan (1976), and Deistler, Dunsmuir and Hannan (1978).

Theorem 2a. Under assumptions 3 and 4, the covariance between $h^{\star}_{n_1,t}$ and $h^{\star}_{n_2,t-m}$, $(n_1, n_2 = 1, \cdots, p^{\star})$ is given by:

$$\gamma^{m}_{n_1,n_2} = cov(h^{\star}_{n_1,t}, h^{\star}_{n_2,t-m}) = \sum_{j=1}^{p^{\star}} \sum_{i=1}^{p^{\star}} \sum_{f=1}^{\bar{p}} \zeta_{fm} \lambda^{n_1 i, n_2 j}_{f,m} \sigma_{ij,v}, \text{ where} \tag{3.8}$$

$$\zeta_{fm} = \frac{B_f^{\bar{p}-1+m}}{\prod\limits_{g=1}^{\bar{p}} (1 - B_f B_g) \prod\limits_{\substack{g=1 \\ g \neq f}}^{\bar{p}} (B_f - B_g)}, \;\; \sigma_{ij,v} = cov(v^{\star}_{it}, v^{\star}_{jt}) \tag{3.8a}$$

$$\lambda^{n_1 i, n_2 j}_{f,m} = \sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{n_1 i}_{d} B^{n_2 j}_{d+c} B^{c}_{f} + \sum_{c=1}^{m^{\star}} \sum_{d=1}^{\tilde{p}-c} B^{n_2 j}_{d} B^{n_1 i}_{d+c} B^{-c}_{f} + \sum_{c=m+1}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{n_2 j}_{d} B^{n_1 i}_{d+c} B^{c-2m}_{f} \tag{3.8b}$$

and $m^{\star} = min(m, \tilde{p})$. The proof follows immediately from the univariate ARMA representations (3.6) and Theorem 1.

Assumption 3a. All roots of $|B^L|$ and all roots of $|B^{L\bullet}|$ lie outside the unit circle. These conditions satisfy the stationarity and invertibility of the model.

Assumption 4a. The polynomials B^L and $B^{L\bullet}$ have no common left factors other than unimodular ones (this condition satisfy the irreducibility of the model) and satisfy other identificiability conditions given in Dunsmuir and Hannan (1976), and Deistler, Dunsmuir and Hannan (1978).

Theorem 2a$'$. Under assumptions 3a and 4a, the covariance between $\epsilon^{\star}_{n_1,t}$ and $\epsilon^{\star}_{n_2,t-m}$ $(n_1,\ n_2 = 1, \cdots, p^{\star})$ is given by

$$\gamma^{m}_{n_1,n_2} = cov(\epsilon^{\star}_{n_1,t} \epsilon^{\star}_{n_2,t-m}) = \sum_{j=1}^{p^{\star}} \sum_{i=1}^{p^{\star}} \sum_{f=1}^{\bar{p}} \zeta_{fm} \hat{\lambda}^{n_1 i, n_2 j}_{f,m} \sigma_{ij,v}, \text{ where} \tag{3.9}$$

$$\hat{\lambda}^{n_1 i, n_2 j}_{f,m} = \sum_{c=0}^{\hat{p}} \sum_{d=0}^{\hat{p}-c} A^{n_1 i}_{d} A^{n_2 j}_{d+c} B^{c}_{f} + \sum_{c=1}^{m^{\star}} \sum_{d=0}^{\hat{p}-c} A_{d} A_{d+c} B^{-c}_{f} + \sum_{c=m+1}^{\hat{p}} \sum_{d=1}^{\hat{p}-c} A^{n_2 j}_{d} A^{n_1 i}_{d+c} B^{c-2m}_{f} \tag{3.9a}$$

and $m^{\star} = min(m, \hat{p})$. The proof follows immediately from the univariate representations (3.7) and Theorem 1.

Corollary 2b$''$. The covariances between $\epsilon^2_{n_1,t-l_1}$ and $h_{n_1,t-l_2}$ can be derived by using the following relations:

$$E(\epsilon^2_{n_1,t} \epsilon_{n_2,t-m}) = E(h_{n_1,t} \epsilon^2_{n_2,t-m}), \;\; E(h_{n_1,t} h_{n_2,t-m}) = E(\epsilon^2_{n_1,t} h_{n_2,t-m}) \tag{3.10}$$

together with Theorems 2a-2a'.

Alternatively the above covariances can be derived by using Corollaries 2b-2b' (i.e. the univariate ARMA representations of the squared errors and of the conditional variances) and Theorem 1.

3.2 Constant Correlation Representation

Next, we examine the constant conditional correlation Multivariate GARCH(r,s) process of order p [M_cGARCH(r,s,p)].

The general form of the M_cGARCH(r,s,p) model is given by

$$\bar{h}_t = \Omega + \sum_{l=1}^{s} {}_lA\,\bar{\epsilon}_{t-l} + \sum_{l=1}^{r} {}_lB^\star\,\bar{h}_{t-l}, \text{ or } B^{L\star}\bar{h}_t = \Omega + A^L\bar{\epsilon}_t, \text{ where} \tag{3.11}$$

$$\bar{h}_t = \begin{bmatrix} h_{1t} \\ \cdot \\ \cdot \\ \cdot \\ h_{pt} \end{bmatrix}, \quad \bar{\epsilon}_{t-l} = \begin{bmatrix} \epsilon_{1,t-l} \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_{p,t-l} \end{bmatrix}, \quad {}_lA = \begin{bmatrix} a^l_{11} & a^l_{1p} \\ \cdots & \cdots \\ \cdots & \cdots \\ a^l_{p1} & a^l_{pp} \end{bmatrix}, \quad {}_lB^\star = \begin{bmatrix} b^{l\star}_{11} & b^{l\star}_{1p} \\ \cdots & \cdots \\ \cdots & \cdots \\ b^{l\star}_{p1} & b^{l\star}_{pp} \end{bmatrix} \tag{3.11a}$$

and the $B^{L\star}$ and A^L are given by (3.2a)- (3.2b) (where now $p^\star$ is replaced by p). The conditional correlation between the errors is constant:

$$\frac{cov_{t-1}(\epsilon_{it}, \epsilon_{jt})}{\sqrt{var_{t-1}(\epsilon^2_{it})var_{t-1}(\epsilon^2_{jt})}} = \frac{h_{ij,t}}{h^{.5}_{it}h^{.5}_{jt}} = p_{ij} \tag{3.11b}$$

Corollary 2c. The VARMA representation of the conditional variances for the M_cGARCH(r,s,p) model is given by

$$\bar{h}_t = \Omega + \sum_{l=1}^{r'} {}_lB\,\bar{h}_{t-l} + \sum_{l=1}^{s} {}_lA\,\bar{v}_{t-l}, \text{ or } B^L\bar{h}_t = \Omega + A^L\bar{v}_t \;\; r' = max(r,s) \tag{3.12}$$

where

$${}_lB = \begin{cases} {}_lB^\star + {}_lA & \text{if } \; l < r, s \\ {}_lB^\star & \text{if } \; l, r > s \\ {}_lA & \text{if } \; l, s > r \end{cases}, \quad \bar{v}_{t-l} = \begin{bmatrix} v_{1t-l} \\ \cdot \\ \cdot \\ \cdot \\ v_{p,t-l} \end{bmatrix} \tag{3.12a}$$

and B^L is as in (3.4a) (where now $p^\star$ is replaced by p).

The v_{it}'s are uncorrelated although not independent terms and are given by $v_{it} = \epsilon_{it}^2 - h_{it}$.

Their covariance matrix is denoted by[14]

$$\bar{\sigma}_v = \begin{bmatrix} \sigma_{11,v} & \sigma_{1p,v} \\ \cdots & \cdots \\ \cdots & \cdots \\ \sigma_{p1,v} & \sigma_{pp,v} \end{bmatrix} \tag{3.12b}$$

Proof. In (3.11) we add and subtract $\sum_{l=1}^{s} {}_l A\, h_{t-l}$ and we get (3.12).

Corollary 2d. The univariate ARMA representations of (3.11) are given by

$$\prod_{l=1}^{\bar{p}} (1 - B_l L) h_{it} = \sum_{j=1}^{p} \sum_{l=1}^{\tilde{p}} B_l^{ij} L^l v_{jt}, \;\; \bar{p} = p \times r', \;\; \tilde{p} = (p-1)r' + s \tag{3.13}$$

where the B_l's and B_l^{ij}'s are as in (3.6) (where now $p^\star$ is replaced by p). The proof is similar to that of the MGARCH(r,s,p) model.

Theorem 2b. Under assumptions 3 and 4, the covariance between $h_{n_1,t}$ and $h_{n_2,t-m}$, $(n_1, n_2 = 1, \cdots, p)$ is given by[15]*:*

$$\gamma_{n_1,n_2}^m = cov(h_{n_1,t}, h_{n_2,t-m}) = \sum_{j=1}^{p} \sum_{i=1}^{p} \sum_{f=1}^{\bar{p}} \zeta_{fm} \lambda_{f,m^\star}^{n_1 i, n_2 j} \sigma_{ij,v}, \;\; \textit{where} \tag{3.14}$$

$$\zeta_{fm} = \frac{B_f^{\bar{p}-1+m}}{\prod_{g=1}^{\bar{p}} (1 - B_g B_f) \prod_{\substack{g=1 \\ g \neq f}}^{\bar{p}} (B_f - B_g)} \tag{3.14a}$$

$$\lambda_{f,m}^{n_1 i, n_2 j} = \sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B_d^{n_1 i} B_{d+c}^{n_2 j} B_f^c + \sum_{c=1}^{m^\star} \sum_{d=1}^{\tilde{p}-c} B_d^{n_2 j} B_{d+c}^{n_1 i} B_f^{-c} + \sum_{c=m+1}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B_d^{n_2 j} B_{d+c}^{n_1 i} B_f^{c-2m} \tag{3.14b}$$

and $i, j, n_1, n_2 = 1, \cdots, p$.

The proof follows immediately from the univariate ARMA representations (3.13) and Theorem 1.

[14] The derivation of the $\sigma_{ij,v}$'s is given in Appendix B.

[15] The derivation of the covariances between the squared errors is omitted since it is similar to that of the vec representation.

4 Concluding Remarks

Since the observed volatilities of asset returns are regarded as realisations of the underlying stochastic processes it is not surprising that so much effort has been lavished on building models to measure and forecast them. The univariate and multivariate GARCH models and its various generalisations have been very popular in this respect and have been applied to various sorts of economic and financial data sets. However, they have seen relatively fewer theoretical advancements. This paper has contributed to the theoretical developments in the multivariate GARCH literature. In Section 2 we presented the autocovariance function of the S-GARCH model. Moreover, in Section 3 we presented the autocovariance function of the vec (note that the vec representation includes the diagonal and the BEKK representations as special cases) and of the constant correlation representations of the multivariate GARCH model. The techniques used in this paper (i.e. the Wold representation and the autocovariance generating function of the univariate ARMA representations of the conditional variances and of the squared errors) can be applied to obtain the covariance structure of (i) more complex univariate GARCH models like the component GARCH and the GARCH-M-X models[16] (see K, 1999d) and (ii) the multivariate GARCH in mean models[17].

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[16] The component GARCH model was introduced by Ding and Granger (1996) whereas the GARCH-M-X model was introduced by Longstaff and Schwartz (1992).

[17] The MGARCH models are discussed by, for example, Song (1996) and Grier and Perry (1996).

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Appendix

A PROOF OF COROLLARY 1b, THEOREM 1

From (2.1) and (2.3) we get

$$h_t = V_{t-1}(y_t) = V_{t-1}(\sum_{i=1}^{n} \epsilon_{it}) = E_{t-1}[\sum_{i=1}^{n} \epsilon_{it}^2 + 2\sum_{k=l+1}^{n}\sum_{l=1}^{n-1} \epsilon_{lt}\epsilon_{kt}] =$$

$$\sum_{i=1}^{n} h_{it} + 2\sum_{k=l+1}^{n}\sum_{l=1}^{n-1} h_{lk,t} = \sum_{i=1}^{n} h_{it} + 2\sum_{k=l+1}^{n}\sum_{l=1}^{n-1} \omega_{lk,0} = \omega_0^\star + \sum_{i=1}^{n} h_{it} \tag{A.1}$$

IMA

The infinite moving average (ima) representations of the h_{it}'s are given by[18]

$$h_{it} = \sum_{r=1}^{\infty} e_i^r v_{it-r}, \text{ where } e_i^r = \sum_{l=1}^{p_i^\star}\sum_{k=1}^{min(r,q_i)} \zeta_{il}^{r-k} a_{ik}, \ \zeta_{il}^m = \frac{(\beta'_{il})^{p_i^\star - 1 + m}}{\prod\limits_{\substack{k=1\\k\neq l}}^{p_i^\star} (\beta'_{il} - \beta'_{ik})} \tag{A.2}$$

From the preceding equation we have

$$cov(h_{it}, h_{j,t-m}) = \sum_{r=1}^{\infty} e_j^r e_i^{r+m} \sigma_{ij,v} \tag{A.2a}$$

After some algebra we get

$$cov(h_{it}, h_{j,t-m}) = (\sum_{l=1}^{p_i^\star} \zeta_{il}^m s_{j,il}^0 \lambda_{l,m}^{ij} + \sum_{k=1}^{p_j^\star} \zeta_{jk}^m s_{i,jk}^0 \lambda_{ij}^{k,m})\sigma_{ij,v}, \text{ where } s_{j,il}^0 = \sum_{k=1}^{p_j^\star} \zeta_{jk}^{0,il}, \tag{A.2b}$$

$$\zeta_{jk}^{0,il} = \frac{\zeta_{jk}^0}{(1 - \beta'_{il}\beta'_{jk})}, \ \lambda_{l,m}^{ij} = \sum_{c=0}^{q_j}\sum_{d=1}^{q'_i} a_{id}a_{j,d+c}(\beta'_{il})^c + \sum_{c=1}^{m^\star}\sum_{d=1}^{q'_j} a_{jd}a_{i,d+c}(\beta'_{il})^{-c} \tag{A.2c}$$

$$\lambda_{ij}^{k,m} = \sum_{c=m+1}^{q_i}\sum_{d=1}^{q'_j} a_{jd}a_{id+c}(\beta'_{jk})^{c-2m}, \tag{A.2d}$$

[18]The ima representation of a GARCH model is given in K(1999a).

and $q'_i = min(q_i, q_j - c)$, $q'_j = min(q_j, q_i - c)$, $m^\star = min(m, q_i)$.
In the preceding equation, we use

$$s^0_{j,il} = \frac{1}{\prod_{k=1}^{p^\star_j}(1-\beta'_{il}\beta'_{jk})} \tag{A.2e}$$

to get

$$cov(h_{it}, h_{j,t-m}) = (\sum_{l=1}^{p^\star_i}\zeta^m_{il,j}\lambda^{ij}_{l,m} + \sum_{k=1}^{p^\star_j}\zeta^m_{jk,i}\lambda^{k,m}_{ij})\sigma_{ij,v}, \text{ where } \zeta^m_{il,j} = \frac{\zeta^m_{il}}{\prod_{k=1}^{p^\star_j}(1-\beta'_{il}\beta'_{jk})} \tag{A.2f}$$

Thus, we have

$$h_t = \omega^\star_0 + \sum_{i=1}^{n} h_{it} \Rightarrow cov_m(h_t) = \sum_{j=1}^{n}\sum_{i=1}^{n} cov(h_{it}, h_{j,t-m})$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{l=1}^{p^\star_i}\zeta^m_{il,j}\lambda^{ij}_{l,m} + \sum_{k=1}^{p^\star_j}\zeta^m_{jk,i}\lambda^{k,m}_{ij})\sigma_{ij,v} \tag{A.2g}$$

■

AGF

From (2.3) we get

$$\frac{1}{B^\star_i(z)B^\star_j(z^{-1})} = \frac{1}{\prod_{l=1}^{p^\star_i}(1-\beta'_{il}z)\prod_{l=1}^{p^\star_j}(1-\beta'_{jl}z^{-1})} = \sum_{k=1}^{p^\star_j}\sum_{l=1}^{p^\star_i}\frac{\zeta^0_{il}\zeta_{jk}}{(1-\beta'_{il}z)(1-\beta'_{jk}z^{-1})} \tag{A.3}$$

$$A_i(z)A_j(z^{-1}) = (\sum_{r=1}^{q_i} a_{ir}z^r)(\sum_{r=1}^{q_j} a_{jr}z^{-r}) = \sum_{l=0}^{q_i}\sum_{k=1}^{q'_i} a_{jk}a_{ik+l}z^l + \sum_{l=1}^{q_j}\sum_{k=1}^{q'_i} a_{ik}a_{jk+l}z^{-l} \tag{A.3a}$$

From the preceding equations we have

$$\frac{A_i(z)A_j(z^{-1})}{(1-\beta'_{il}z)(1-\beta'_{jk}z^{-1})} =$$

$$= \frac{\zeta^0_{il}\zeta^0_{jk}}{(1-\beta'_{il}\beta'_{jk})}\sum_{m=0}^{\infty} f_m[(\lambda^{ij}_{l,m}(\beta'_{il})^m + \lambda^{km}_{ij}(\beta'_{jk})^m)z^m + (\lambda^{ji}_{k,m}(\beta'_{jk})^m + \lambda^{l,m}_{ji}(\beta'_{il})^m)z^{-m}] \tag{A.3b}$$

Using

$$\sum_{k=1}^{p^\star_j}\frac{\zeta^0_{jk}}{(1-\beta'_{il}\beta'_{jk})} = \sum_{k=1}^{p^\star_j}\zeta^{0,il}_{jk} = \frac{1}{\prod\limits_{k=1}^{p^\star_j}(1-\beta'_{il}\beta'_{jk})} \tag{A.3c}$$

in (A.3)-(A.3b) we obtain

$$\frac{A_i(z)A_j(z^{-1})}{B^\star_i(z)B^\star_j(z^{-1})} + \frac{A_j(z)A_i(z^{-1})}{B^\star_j(z)B^\star_i(z^{-1})} =$$

$$= \sum_{k=1}^{p^\star_j}\sum_{l=1}^{p^\star_i}\sum_{m=0}^{\infty}\frac{\zeta^0_{il}\zeta^0_{jk}}{(1-\beta'_{il}\beta'_{jk})}[(\lambda^{ij}_{l,m}(\beta'_{il})^m + \lambda^{k,m}_{ij}(\beta'_{jk})^m)$$

$$+ (\lambda^{ji}_{k,m}(\beta'_{jk})^m + \lambda^{l,m}_{ji}(\beta'_{il})^m)](z^m + z^{-m})$$

$$= \sum_{m=0}^{\infty}[(\sum_{l=1}^{p^\star_i}\zeta^m_{il,j}\lambda^{ij}_{l,m} + \sum_{k=1}^{p^\star_j}\zeta^m_{jk,i}\lambda^{k,m}_{ij}) + (\sum_{k=1}^{p^\star_j}\zeta^m_{jk,i}\lambda^{ji}_{k,m} + \sum_{l=1}^{p^\star_i}\zeta^m_{il,j}\lambda^{l,m}_{ji})](z^m + z^{-m}) \tag{A.3d}$$

Using the preceding equation we see that

$$h_t = \omega^\star_0 + \sum_{i=1}^{n} h_{it} \Rightarrow g_z(h) = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{A_i(z)A_j(z^{-1})}{B^\star_i(z)B^\star_j(z^{-1})}\sigma_{ij,v} =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{m=0}^{\infty} f_m[\sum_{l=1}^{p^\star_i}\zeta^m_{il,j}\lambda^{ij}_{l,m} + \sum_{k=1}^{p^\star_j}\zeta^m_{jk,i}\lambda^{ij}_{k,m}]\sigma_{ij,v}(z^m + z^{-m}) \tag{A.3e}$$

■

B *PROOF OF THEOREM 2a*

UNIVARIATE ARMA REPRESENTATIONS

The univariate representations of (3.3) are given by (in what follows B denotes matrix whereas $\bar{B}$ denotes determinant)

$$\sum_{l=0}^{\bar{p}} {}^{p^\star}\bar{B}^\star_{lm}\, L^l h^\star_{it} = \prod_{l=1}^{\bar{p}} (1 - B_l L) h^\star_{it} = \sum_{j=1}^{p^\star} \sum_{l=1}^{\tilde{p}} {}^{p^\star,ij}\bar{B}^\star_{lm}\, L^l v^\star_{jt} =$$
$$= \sum_{j=1}^{p^\star} \sum_{l=1}^{\tilde{p}} B^{ij}_l L^l v^\star_{jt} \tag{B.1}$$

where

$${}^{p^\star}\bar{B}^\star_{lm} = \sum_{m=1}^{p^\star} \Re'_{ml,r'}\, {}^{p^\star}\bar{B}_m, \quad {}^{p^\star}\bar{B}_m = \prod_{k=1}^{m} \Big(\sum_{f_k = f_{k-1}+1}^{p^\star - (m-k)} \Big) \prod_{k=1}^{m} ({}^{p^\star}B^{f_k}_{f_k})(-1)^m, \quad f_0 = 0 \tag{B.1a}$$

where ${}^{p^\star}\bar{B}_m$ denotes the sum of the determinants of all the $(m \times m)$ submatrices of the $(p^\star \times p^\star)$ matrix B. As an example, consider the case where $p^\star = 3$ and $m = 2$:

$${}^{p^\star}\bar{B}_m = {}^3\bar{B}_2 = {}^3\bar{B}^{12}_{12} + {}^3\bar{B}^{13}_{13} + {}^3\bar{B}^{23}_{23} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}$$

$$\Re'_{ml,r'} = \begin{cases} \Re_{ml,r'} & \text{if}\, l = m, \cdots, m \times r' \\ 0 & \text{otherwise} \end{cases}, \quad \Re_{ml,r'} = \prod_{k=1}^{m} \mho_{g_k = max[1, l - [(m-k)r' + \sum_{t=1}^{k-1} g_t]]}^{min[l - \sum_{t=1}^{k-1} g_t - (m-k), r']} g_k$$

where $\Re_{ml,r'}$ denotes the set of all the combinations of m numbers taking values from 1 to r' and adding to l. As an example, consider the case where $r' = 2$ and $m = 2$:

$$\Re'_{ml,r'} = \Re'_{2l,2} = \begin{cases} \Re_{2l,2} & \text{if } l = 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}, \quad \Re_{22,2} = 11, \;\; \Re_{23,2} = 12, 21, \;\; \Re_{24,2} = 22$$

${}^{p^\star}_{k_1 k_2 \cdots k_m}\bar{B}_m$ $(k_i = 1, \cdots, r')$ denotes the ${}^{p^\star}\bar{B}_m$ sum of determinants where now the b's in the ith column $(i = 1, \cdots, m)$ are taken from the ${}_{k_i}B$ matrix. As an example, consider the case where $p^\star = 3$ and $m = 3$:

$${}^{3}_{121}\bar{B}_3 = - \begin{vmatrix} b^1_{11} & b^2_{12} & b^1_{13} \\ b^1_{21} & b^2_{22} & b^1_{23} \\ b^1_{31} & b^2_{32} & b^1_{33} \end{vmatrix}$$

When we multiply ${}^{p^\star}\bar{B}_m$ by $\Re_{ml,r'}$ we get

$$ {}^{p^\star}_{(l_1\cdots l_m)_1}\bar{B}_m + \cdots + {}^{p^\star}_{(l_1\cdots l_m)_f}\bar{B}_m $$

where $(l_1 \cdots l_m)_f$ denotes the set of all the f different combinations of m numbers which take values from 1 to $p^\star$ and sum to l. As an example, consider the case where $p^\star = 3$ and $m = 2,\ l = 3$:

$$ \Re_{23,2}\, {}^{3}\bar{B}_2 = {}^{3}_{12}\bar{B}_2 + {}^{3}_{21}\bar{B}_2 = [\begin{vmatrix} b^1_{11} & b^2_{12} \\ b^1_{21} & b^2_{22} \end{vmatrix} + \begin{vmatrix} b^1_{11} & b^2_{13} \\ b^1_{31} & b^2_{33} \end{vmatrix} + \begin{vmatrix} b^1_{22} & b^2_{23} \\ b^1_{32} & b^2_{33} \end{vmatrix}] + $$

$$ +[\begin{vmatrix} b^2_{11} & b^1_{12} \\ b^2_{21} & b^1_{22} \end{vmatrix} + \begin{vmatrix} b^2_{11} & b^1_{13} \\ b^2_{31} & b^1_{33} \end{vmatrix} + \begin{vmatrix} b^2_{22} & b^1_{23} \\ b^2_{32} & b^1_{33} \end{vmatrix}] $$

$$ {}^{p^\star,ij_1}\bar{B}^\star_{lm} = \sum_{m=0}^{p^\star-1} \Re'_{(m+1)l,r's}\, {}^{p^\star,ij_1}\bar{B}_m, \quad {}^{p^\star,ij_1}\bar{B}_m = \prod_{k=1}^{m} \Big(\sum_{f_k=f_{k-1}+1}^{p^\star-(m-k)} \Big) \prod_{k=1}^{m} ({}^{p^\star,ij_1}B^{1,f_k}_{1,f_k})(-1)^m, \ f_0 = 1 \tag{B.1b} $$

where ${}^{p^\star,ij}B$ denotes a $(p^\star \times p^\star)$ matrix. It is obtained from matrix B by substituting its ith column with the ith column of matrix A. As an example, consider the case where $p^\star = 3,\ i = 3$ and $j = 1$:

$$ {}^{3,31}B = \begin{bmatrix} b_{11} & b_{12} & a_{11} \\ b_{21} & b_{22} & a_{21} \\ b_{31} & b_{32} & a_{31} \end{bmatrix} $$

${}^{p^\star,ij_1}B$ denotes a $(p^\star \times p^\star)$ matrix. It is obtained from matrix ${}^{p^\star,ij}B$ by moving the ith row (column) into the first row (column). As an example, consider the case where $p^\star = 3,\ i = 3$ and $j = 1$:

$$ {}^{3,31_1}B = \begin{bmatrix} a_{31} & b_{31} & b_{32} \\ a_{11} & b_{11} & b_{12} \\ a_{21} & b_{21} & b_{22} \end{bmatrix} $$

Of all the $(m+1) \times (m+1)$ submatrices of the $(p^\star \times p^\star)$ matrix ${}^{p^\star,ij_1}B$, ${}^{p^\star,ij_1}B_m$ denotes the sum of those which include elements of its first row and column. As an example, consider the case where $p^\star = 3,\ i = 3,\ j = 1$ and $m = 1$:

$$ {}^{3,31_1}B_1 = \begin{bmatrix} a_{31} & b_{31} \\ a_{11} & b_{11} \end{bmatrix} + \begin{bmatrix} a_{31} & b_{32} \\ a_{21} & b_{22} \end{bmatrix} $$

$$ \Re'_{(m+1)l,r's} = \begin{cases} \Re'_{(m+1)l,r's} & \text{if } l = m+1, \cdots, mr' + s \\ 0 & \text{otherwise} \end{cases} $$

$$ \Re_{(m+1)l,r's} = \prod_{k^\star=1}^{m+1} \mho_{g_{k^\star}=max[1,l-[(m+1-k^\star)r^\star + \sum_{t=1}^{k^\star-1} g_t]]}^{min[l-\sum_{t=1}^{k^\star-1} g_t - (m+1-k^\star), r^\star]} g_{k^\star}, \quad r^\star = \begin{cases} s & \text{if } k^\star = 1 \\ r' & \text{otherwise} \end{cases} $$

$\Re_{(m+1)l,r's}$ denotes the set of all the combinations of $(m+1)$ numbers which take values from 1 to r' (except from the first one which take values from one to s) and sum to l. As an example, consider the case where $r' = 2,\ \ s = 3,\ \ l = 6$ and $m = 2$:

$$\Re_{36,23} = 222,\ 312,\ 321$$

The proof is given in K(1999c). ■

The covariance matrix $\bar{\sigma}_v^\star$

Using Theorem 1 and expression (B.1) we can show that

$$h'_{it} - h^\circ_{it} = \sum_{j=1}^{p'} \gamma_{ij,0} v'_{j,t}, \text{ for } i = 1, \cdots, p' \text{ where} \tag{B.2}$$

$\underline{h'_{it}}$

The h'_{it}'s are given by

$$h'_{it} = \begin{cases} E(h^2_{it}) & \text{if } i \leq p \\ E(h^2_{nk,t}) & \text{if } p^\star \geq i > p,\ \ i = n \cdot p + k - \frac{n(n+1)}{2},\ \ \begin{array}{rcl} n & = & 1, \cdots, p-1 \\ k & = & n+1, \cdots, p \end{array} \\ E(h_{n^\star t} h_{k^\star t}) & \text{if } p' \geq i > p^\star,\ \ i = n^\star p^\star + k^\star - \frac{n^\star(n^\star+1)}{2},\ \ \begin{array}{rcl} n^\star & = & 1, \cdots, p^\star - 1 \\ k^\star & = & n^\star + 1, \cdots, p^\star \end{array} \end{cases} \tag{B.2a}$$

and the $n^\star$ and $k^\star$ are given by

$$n^\star = \begin{cases} n^\star & \text{if } n^\star \leq p, \\ l_n \cdot m_n & \text{if } n^\star > p,\ \ n^\star = l_n p + m_n - \frac{l_n(l_n+1)}{2}. \end{cases}$$

$$k^\star = \begin{cases} k^\star & \text{if } k^\star \leq p, \\ l_k \cdot m_k & \text{if } k^\star > p,\ \ l_k p + m_k - \frac{l_k(l_k+1)}{2} \end{cases} \tag{B.2b}$$

$$\begin{array}{rclrcl} l_n & = & 1, \cdots, p-2, & l_k & = & 1, \cdots, p-1 \\ m_n & = & l_n + 1, \cdots, p, & m_k & = & l_k + 1, \cdots, p \end{array}$$

$\underline{h^\circ_{it}}$

The h°_{it}'s are given by

$$h^{\circ}_{it} = \begin{cases} [E(h_{it})]^2 & \text{if } i \leq p, \\ [E(h_{nk,t})]^2 & \text{if } p^{\star} \geq i > p \\ E(h_{n^{\star}t})E(h_{k^{\star}t}) & \text{if } p' \geq i > p^{\star}, \end{cases} \text{ and } E(h_{it}) = \frac{\omega^{\star}_i}{B(1)} \tag{B.2c}$$

The $n^{\star}$ and $k^{\star}$ are given by (B.2a)-(B.2b).

$\underline{\gamma_{ij,0}}$

a) $p^{\star} \geq i \geq 1$

When $p^{\star} \geq i$ the $\gamma_{ij,0}$'s are given by

$$\gamma_{ij,0} = \begin{cases} \sum_{f=1}^{\tilde{p}} \zeta_{f0} \lambda^{i,j}_{f,0} & \text{if } j \leq p^{\star}, \\ 2\sum_{f=1}^{\tilde{p}} \zeta_{f0} \lambda^{i,n^{\star}k^{\star}}_{f,0} & \text{if } j > p^{\star}, \ j = n^{\star}p^{\star} + k^{\star} - \frac{n^{\star}(n^{\star}+1)}{2} \end{cases} \tag{B.2d}$$

$$\lambda^{i,j}_{f0} = 2\sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{ij}_d B^{ij}_{d+c} (B_f)^c, \ \ \lambda^{i,n^{\star}k^{\star}}_{f0} = \sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{in^{\star}}_d B^{ik^{\star}}_{d+c} (B_f)^c + \sum_{c=1}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{ik^{\star}}_d B^{in^{\star}}_{d+c} (B_f)^c$$

$$\zeta_{f0} = \frac{B_f^{\tilde{p}-1}}{\prod_{g=1}^{\tilde{p}} (1 - B_f B_g) \prod_{\substack{g=1 \\ g \neq f}}^{\tilde{p}} (B_f - B_g)} \tag{B.2d'}$$

b) $p' \geq i > p^{\star}$

when $p' \geq i > p^{\star}$, $i = n'p^{\star} + k' - \frac{n'(n'+1)}{2}$, $n' = 1, \cdots, p^{\star} - 1$, $k' = n' + 1, \cdots, p^{\star}$ the $\gamma_{ij,0}$'s are given by

$$\gamma_{ij,0} = \begin{cases} \sum_{f=1}^{\tilde{p}} \zeta_{f0} \lambda^{n'k',j}_{f0} & \text{if } j \leq p^{\star} \\ \sum_{f=1}^{\tilde{p}} \zeta_{f0} [\lambda^{n'k',n^{\star}k^{\star}}_{f0} + \lambda^{n'k',k^{\star}n^{\star}}_{f0}] & \text{if } p' \geq j > p^{\star} \end{cases} \tag{B.2e}$$

$$\lambda^{n'k',j}_{f0} = \sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{n'j}_d B^{k'j}_{d+c} (B_f)^c + \sum_{c=1}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{k'j}_d B^{n'j}_{d+c} (B_f)^c \tag{B.2 e'}$$

$$\lambda^{n'k',n^{\star}k^{\star}}_{f0} = \sum_{c=0}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{n'n^{\star}}_d B^{k'k^{\star}}_{d+c} B_f^c + \sum_{c=1}^{\tilde{p}} \sum_{d=1}^{\tilde{p}-c} B^{k'k^{\star}}_d B^{n'n^{\star}}_{d+c} (B_f)^c \tag{B.2e''}$$

The $n^{\star}$, and $k^{\star}$ are as in (B.2a)-(B.2b).

$\underline{v'_{jt}}$

Multivariate Normal Distribution

From the definition of the Multivariate ARCH model, proposed by EGK (1984) we have

$$\epsilon_{it} = h_{it}^{1/2} e_{it}, \;\; i = 1, \cdots, p \tag{B.3}$$

$$E_{t-1}(e_{it} e_{jt}) = p_{ij} = \frac{h_{ij,t}}{h_{it}^{1/2} h_{jt}^{1/2}} \tag{B.3a}$$

Using the moment generating function of the multivariate normal distribution we obtain the moments of the e_{it}'s[19]:

$$\begin{aligned} E_{t-1}(e_{it}^2 e_{jt}^2) &= 1 + 2p_{ij}^2 & E_{t-1}(e_{it}^3 e_{jt}) &= 3p_{ij} \\ E_{t-1}(e_{it}^2 e_{jt} e_{nt}) &= p_{jn} + 2p_{ij} p_{in} & E_{t-1}(e_{it} e_{jt} e_{nt} e_{kt}) &= p_{ij} p_{nk} + p_{in} p_{jk} + p_{ik} p_{jn} \end{aligned} \tag{B.3b}$$

Inserting the preceding equations in (3.3b) and using (B.3a) we get

$$\begin{aligned} var(v_{it}) &= 2E(h_{i,t}^2) & var(v_{ij,t}) &= E(h_{it} h_{jt}) + E(h_{ij,t}^2) \\ cov(v_{it}, v_{jt}) &= 2E(h_{ij,t}^2) & cov(v_{it}, v_{ij,t}) &= 2E(h_{i,t} h_{ij,t}) \\ cov(v_{it}, v_{jn,t}) &= 2E(h_{ij,t} \cdot h_{in,t}) \\ cov(v_{ij,t}, v_{in,t}) &= E(h_{it} \cdot h_{jn,t}) + E(h_{ij,t} \cdot h_{in,t}) \\ cov(v_{ij,t}, v_{nk,t}) &= E(h_{ik,t} \cdot h_{jn,t}) + E(h_{in,t} \cdot h_{jk,t}) \end{aligned} \tag{B.4}$$

In what follows, using the preceding equations (B.4) we express the $v'_{j,t}$'s as functions of the h'_{it}'s:

i) $p^\star \geq j$

When $p^\star \geq j$ then the v'_{jt}'s are given by

$$v'_{jt} = var(v_{j,t}) = \begin{cases} 2E(h_{j,t}^2) = 2h'_{j,t} & \text{if } j \leq p, \\ var(v_{nk,t}) \;=\; E(h_{n,t} \cdot h_{k,t}) + E(h_{nk,t}^2) = \\ \qquad\qquad\;\; = \; h'_{j^\star,t} + h'_{jt} & , \;\; \text{if } p^\star \geq j > p \end{cases} \tag{B.5}$$

where

$$j = n \cdot p + k - \frac{n(n+1)}{2}, \qquad n = 1, \cdots, p-1$$
$$k = n+1, \cdots, p, \qquad j^\star = np^\star + k - \frac{n(n+1)}{2}$$

[19]For the properties of the multivariate normal distribution see Kendall and Stuart (1977, pp. 372-392), Muirhead (1982, pp. 2-20) or Anderson (1984, pp. 6-50).

ii) $j > p^\star$

If $j > p^\star$, where

$$j = n^\star p^\star + k^\star - \frac{n^\star(n^\star+1)}{2}, \;\; n^\star = 1, \cdots, p^\star - 2, \;\; k^\star = n^\star + 1, \cdots, p^\star,$$

we can distinguish three cases:

a) $n^\star \leq p \quad k^\star \leq p$

If $n^\star \leq p, \;\; k^\star \leq p$, then the $v'_{j,t}$'s are given by

$$v'_{jt} = cov(v_{n^\star,t}, v_{k^\star,t}) = 2E(h^2_{n^\star k^\star,t}) = 2h'_{j^\star,t} \tag{B.5a}$$

where $j^\star = n^\star p + k^\star - \frac{n^\star(n^\star+1)}{2}$,

b) $n^\star \leq p, \quad k^\star > p$

If $n^\star \leq p$, $k^\star > p$, where

$$k^\star = l_k p + m_k - \frac{l_k(l_k+1)}{2}, \;\; l_k = 1, \cdots, p-1, \;\; m_k = l_k + 1, \cdots, p$$

then the $v'_{j,t}$'s are given by

$$v'_{jt} = cov(v_{n^\star,t}, v_{l_k m_k,t}) = 2E(h_{n^\star,t} \cdot h_{n^\star m_k,t}) = 2h'_{j^\star,t}$$

$$j^\star = n^\star(p^\star + p) + m_k - n^\star(n^\star + 1) \tag{B.5b}$$

(when $l_k = n^\star$)

$$v'_{jt} = cov(v_{n^\star,t}, v_{l_k m_k,t}) = 2E(h_{n^\star l_k,t} \cdot h_{n^\star m_k,t}) = 2h'_{j^\star,t} \qquad j^\star = n' p^\star + k' - \frac{n'(n'+1)}{2}$$

$$n' = n^\star p + l_k - \frac{n^\star(n^\star+1)}{2} \qquad k' = n^\star p + m_k - \frac{n^\star(n^\star+1)}{2} \tag{B.5c}$$

(when $l_k, \; m_k \neq n^\star$)

c) $n^\star > p$, $k^\star > p$

If $n^\star > p$, $k^\star > p$, where

$$n^\star = l_n p + m_n - \frac{l_n(l_n+1)}{2} \qquad l_n, \;\; l_k = 1, \cdots, p-2$$

$$k^\star = l_k p + m_k - \frac{l_k(l_k+1)}{2} \qquad m_n = l_n + 1, \cdots, p$$

$$m_k = l_k + 1, \cdots, p$$

then the v'_{jt}'s are given by

$$v'_{jt} = cov(v_{l_n m_n,t}, v_{l_k m_k,t}) = E(h_{l_n l_k,t} \cdot h_{m_n m_k,t}) + E(h_{l_n m_k,t} \cdot h_{m_n l_k,t}) = h'_{j^\star,t} + h'_{j^\circ,t} \tag{B.5d}$$

(when $l_n,\ m_n \neq l_k,\ m_k$)
where

$$\begin{aligned} j^\star &= n^\circ p^\star + k^\circ - \frac{n^\circ(n^\circ+1)}{2} & n^\circ &= \min(n', k'),\ \ k^\circ = \max(n', k') \\ n' &= l'_n p + m'_n - \frac{l'_n(l'_n+1)}{2} & k' &= l'_k p + m'_k - \frac{l'_k(l'_k+1)}{2} \end{aligned}$$

$$v'_{jt} = cov(v_{l_n m_n,t}, v_{l_n m_k,t}) = E(h_{l_n,t} \cdot h_{m_n m_k,t}) + E(h_{l_n m_n,t} \cdot h_{l_n m_k,t}) = h'_{j^\star,t} + h'_{j^\circ,t} \tag{B.5e}$$

(when $l_k = l_n$)
where

$$\begin{aligned} j^\star &= l_n p^\star + k' - \frac{l_n(l_n+1)}{2} & j^\circ &= n^\circ p^\star + k^\circ - \frac{n^\circ(n^\circ+1)}{2} \\ k' &= l'_k p + m'_k - \frac{l'_k(l'_k+1)}{2} & n^\circ &= l'_n p + m'_n - \frac{l'_n(l'_n+1)}{2} \\ k^\circ &= l^\circ_k p + m^\circ_k - \frac{l^\circ_k(l^\circ_k+1)}{2} & l'_k &= \min(m_n,\ m_k) \\ m'_k &= \max(m_n,\ m_k) & l'_n &= \min(l_n,\ m_n) \\ m'_n &= \max(l_n,\ m_n) & l^\circ_k &= \min(l_n, m_k) \\ m^\circ_k &= \max(l_n, m_k) & & \end{aligned}$$

Having the v'_{jt}'s as functions of the h'_{it}'s we substitute (B.5)-(B.5e) into (B.2) and we express the h'_{it}'s as functions of the h°_{it}'s:

$$\Gamma' \cdot \bar{h}'_t = \bar{h}^\circ_t \Rightarrow \bar{h}'_t = \Gamma^\star \cdot \bar{h}^\circ_t \Rightarrow h'_{it} = \sum_{j=1}^{p'} \gamma^\star_{ij} h^\circ_{jt}, \text{ where} \tag{B.6}$$

Γ' is a $p' \times p'$ matrix. It's mnth element is γ'_{mn}.
$\bar{h}'_t$ is a $p' \times 1$ vector matrix. It's m1th element is h'_{mt}.
$\bar{h}^\circ_t$ is a $p' \times 1$ vector matrix. It's m1th element is h°_{mt}.
Using Theorem 1 and expression (B.1) we get (3.8) where the $\sigma_{ij,v}$'s are given by (B.4).
■

The covariance matrix $\bar{\sigma}_v$

Using Theorem 1 and expression (3.13) we obtain

$$h'_{it} - h^{\circ}_{it} = \sum_{j=1}^{p^\star} \gamma_{ij,0} v'_{j,t-1}, \quad i = 1, \cdots, p^\star \tag{B.7}$$

$\underline{h'_{it}}$

The h'_{it}'s are given by

$$h'_{it} = \begin{cases} E(h^2_{it}) & \text{if } i \leq p, \\ E(h_{nt} \cdot h_{kt}) & \text{if } p^\star \geq i > p. \end{cases} \tag{B.7a}$$

where

$$i = n \cdot p + k - \frac{n(n+1)}{2}, \qquad \begin{aligned} & n = 1, \cdots, p-1 \\ & k = n+1, \cdots, p \end{aligned}$$

$\underline{h^{\circ}_{it}}$

The h°_{it}'s are given by

$$h^{\circ}_{it} = \begin{cases} [E(h_{it})]^2 & \text{if } i \leq p \\ E(h_{nt})E(h_{kt}) & \text{if } p^\star \geq i > p, \ \text{ and } \ E(h_{it}) = \frac{\omega^\star_i}{B(1)} \end{cases} \tag{B.7b}$$

$\underline{\gamma_{ij,0}}$

a) $p \geq i \geq 1$

When $p \geq i \geq 1$ the $\gamma_{ij,0}$'s are given by

$$\gamma_{ij} = \begin{cases} \sum_{f=1}^{p} \zeta_{f0} \lambda^{j_i, j_i}_{f,0} & \text{if } j \leq p, \\ 2 \sum_{f=1}^{p} \zeta_{f0} \lambda^{n_i k_i}_{f,0} & \text{if } p^\star \geq j > p. \end{cases} \tag{B.7c}$$

where

$$j = n^\star p + k^\star - \frac{n^\star(n^\star+1)}{2}, \ \ n^\star = 1, \cdots, p-1, \ \ k^\star = 1+n, \cdots, p$$

and λ^{j_i, j_i}_{f0}, λ^{n_i, k_i}_{f0} and ζ_{f0} are as in (3.8a-3.8b),

b) $p^\star \geq i > p$

When $p^\star \geq i > p$ where

$$i = n'p + k' - \frac{n'(n'+1)}{2}, \; n' = 1, \cdots, p-1, \; k' = n'+1, \cdots, p$$

the $\gamma_{ij,0}$'s are given by

$$\gamma_{ij,0} = \begin{cases} \sum_{f=1}^{\bar{p}} \zeta_{f0} \lambda_{f0}^{n'k',j} & \text{if } p \geq j \geq 1 \\ \sum_{f=1}^{\bar{p}} \zeta_{f0} [\lambda_{f0}^{n'k',n^\star k^\star} + \lambda_{f0}^{n'k',k^\star n^\star}] & \text{if } p^\star \geq j > p \end{cases} \qquad (B.7d)$$

where the $\lambda_{f0}^{n'k',j}$ and $\lambda_{f0}^{n'k',n^\star k^\star}$ are as in (B.2e'- B.2e'').

Moreover, from the definition of the constant conditional correlation Multivariate GARCH model, proposed by B (1990), we have

$$\epsilon_{it} = h_{it}^{1/2} e_{it}, \;\; E_{t-j}(e_{it} e_{jt}) = p_{ij}, \;\; i = 1, \cdots, p \qquad (B.8)$$

Using the moment generating function of the multivariate normal distribution we obtain the moments of the e_{it}'s:

$$E_{t-1}(e_{it}^2 e_{jt}^2) = 1 + 2p_{ij}^2 \qquad (B.8a)$$

Using the preceding equation and the definition of the v_{it}'s we get

$$cov(v_{it}, v_{jt}) = 2p_{ij}^2 E(h_{i,t} h_{j,t}) \qquad (B.9)$$

Using the preceding equation, we obtain the $v'_{j,t}$'s

$$v'_{j,t} = \begin{cases} var(v_{j,t}) = 2E(h_{j,t}^2) = 2h'_{jt} & \text{if } j \leq p, \\ cov(v_{n,t}, v_{k,t}) = 2p_{nk}^2 E(h_{nt} \cdot h_{kt}) = 2p_{nk}^2 h'_{jt} & \text{if } p^\star \geq j > p. \end{cases} \qquad (B.10)$$

where

$$j = np + k - \frac{n(n+1)}{2}, \; n = 1 \cdots, p-1, \;\; k = n+1, \cdots, p$$

Inserting the preceding equation in (B.7) we obtain

$$,' \cdot \bar{h}'_t = \bar{h}^\circ_t \Rightarrow \bar{h}'_t = ,^\star \cdot \bar{h}^\circ_t \Rightarrow h'_{it} = \sum_{j=1}^{p^\star} \gamma^\star_{ij} h^\circ_{jt}, \text{ where} \qquad (B.11)$$

$,'$ is a $p^\star \times p^\star$ matrix. It's mnth element is γ'_{mn}.
$\bar{h}'_t$ is a $p^\star \times 1$ vector matrix. It's m1th element is h'_{mt}.
$\bar{h}^\circ_t$ is a $p^\star \times 1$ vector matrix. It's m1th element is h°_{mt}.
Using Theorem 1 and expression (3.13) we get (3.14) where the $\sigma_{ij,v}$'s are given by (B.10-B.11).

■