in the general case to obtain a *pair* of equivalent observations. It will be noticed that the parameters we have been concerned with are parameters of groups acting transitively on the space of the observations; when we come to generalise the theory to the greatest possible extent, without introducing concepts such as 'approximate equivalence' of observations, we find that something like this group property is essential.

Reference

1. R. A. Fisher, Contributions to mathematical statistics, Paper 24, *Proc. R. Soc.* A, **144**, 303–305 (1934).

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The 17 plane symmetry groups

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Several modern mathematics courses contain a description of the 17 distinct 'wallpaper patterns'. Others contain the definition of "group" and "isomorphism", together sometimes with a vague statement that these concepts can be used to justify the fact that there are precisely 17 patterns. But neither the passive contemplation of wallpaper patterns, nor the passive contemplation of abstract definitions, is mathematics: the latter is above all an activity in which definitions are used to obtain concrete results. For this reason I have often been asked by teachers what is needed to give a rigorous proof that there are precisely 17 patterns. This article is an exposition of such a proof which is on the one hand elementary, using only basic properties of groups and well known facts about rotations and reflections, but which on the other hand uses methods which are not special to the plane and can be applied also in higher dimensions. The proof suffers from one great defect: there are no pictures. This is intended to emphasise the fact that the proof is a rigorous piece of group theory, and also to persuade the reader to draw pictures of typical patterns to illustrate the central ideas for himself.

1. Symmetry groups

The elements of a symmetry group G will be written in the form (v, φ) where $v \in \mathbb{R}^2$ is a translation vector and φ is a linear transformation of $\mathbb{R}^2$ which preserves distance. We assume as known the fact that such a linear transformation is either a rotation about the origin or a reflection in a line through the origin. By definition, the symmetry (v,φ) sends a point x to

$$(v,\varphi)x = v + \varphi x.$$

Group multiplication in G is composition of symmetries; thus

$$((v,\varphi).(w,\psi))x = (v,\varphi)(w+\psi x) = v + \varphi w + \varphi\psi x$$

and therefore

$$(v,\varphi).(w,\psi) = (v + \varphi w, \varphi\psi). \tag{$*$}$$

We use this formula to obtain four invariants of the group G.

(i) *The lattice T.* Let ι denote the identity linear transformation. By $(*)$ we have $(s,\iota).(t,\iota) = (s+t,\iota)$, and therefore the set

$$T = \{t \in \mathbb{R}^2 : (t,\iota) \in G\}$$

of translation vectors which are also symmetries in G may be regarded both as a subgroup of $\mathbb{R}^2$ (by addition of translation vectors) and as a subgroup of G (by composition of symmetries).

(ii) *The point group H.* By $(*)$ the set

$$H = \{\varphi : (v,\varphi) \in G \text{ for some } v \in \mathbb{R}^2\}$$

is a group. Note that every $\varphi \in H$ is either a rotation or a reflection, and that the set of all rotations in H is a subgroup H_0 of H.

(iii) *The action of H on T.* If $t \in T$ and $\varphi \in H$ then by $(*)$

$$\begin{aligned}(\varphi t,\iota) &= (v + \varphi t - v, \varphi\varphi^{-1}) = (v+\varphi t,\varphi).(-\varphi^{-1}v,\varphi^{-1})\\ &= (v,\varphi).(t,\iota).(v,\varphi)^{-1}\end{aligned}$$

is in G and therefore $\varphi t \in T$. The resulting homomorphism $\varphi : T \to T$ is called the action of φ on T.

(iv) *The shift vectors.* Let $\varphi \in H$ be of finite order q. Then $(v,\varphi) \in G$ for some $v \in \mathbb{R}^2$ and $\varphi^q = \iota$, so by $(*)$

$$(v,\varphi)^q = (v + \varphi v + \ldots + \varphi^{q-1}v, \varphi^q) = (a,\iota),$$

where $a \in T$. The formula shows that $\varphi a = a$. If v' is another choice for v, then

$$(v,\varphi).(v',\varphi)^{-1} = (v,\varphi).(-\varphi^{-1}v',\varphi^{-1}) = (v - v',\iota)$$

and therefore $v - v'$ is a vector $t \in T$. The formula shows that, if v is replaced by v', then a is replaced by $a - (t + \varphi t + \ldots + \varphi^{q-1}t)$. The vectors which arise in this way are called the shift vectors of φ.

Remark. In the remaining paragraphs, properties of the four invariants are established in sections correspondingly numbered (i), (ii), (iii), (iv). The aim is to prove that these four invariants are actually sufficient to classify the groups G in certain cases. Since the above definitions are somewhat abstract, it may help to keep in mind the following informal description. Think of G as the group of symmetries of a two-dimensional pattern or of a three-dimensional crystal, and suppose that the pattern or crystal is formed by juxtaposition of identical building blocks placed in parallel. The *lattice* T is obtained by choosing a vector t at the centre of each building block. Some symmetries in G do not merely translate the blocks but also include a rotation or reflection. The *point group* H is the collection of all rotations and reflections φ which occur in this way. The *shift vectors* of φ measure the extent to which φ must be combined with a translation not in T to obtain a symmetry of G. This occurs, for example, if φ is a reflection which contributes to a two-dimensional or three-dimensional *glide transformation*, or if φ is a rotation which contributes to a three-dimensional *screw transformation.*

2. *Plane groups*

In this paragraph we define precisely which symmetry groups will be classified. It is possible to give a weaker definition and then—with some hard work—to deduce that given here as a consequence. The latter is in any case very natural: it corresponds to the idea that T must not be too small (as would be the case if G were the group of all symmetries of a 'rose' or 'frieze' pattern), and that H must not be too large (as would be the case if G were the group of *all* symmetries of the plane).

DEFINITION. G is a *plane group* if there are linearly independent vectors t_1, t_2 such that the lattice of G has the form

$$T = \{n_1 t_1 + n_2 t_2 : n_1, n_2 \text{ integers}\}$$

and if moreover the point group H of G is finite.

This definition has some immediate consequences which limit severely the invariants which can occur.

(i) *The lattice T must contain a non-zero vector t of minimum length $|t|$.* This follows from the definition because, for any $c > 0$, there are only finitely many pairs of integers (n_1, n_2) such that $|n_1 t_1 + n_2 t_2| < c$. Note that t is not unique, since $|t| = |\varphi t|$ for all $\varphi \in H$; we assert merely that there is no $s \in T$ with $0 < |s| < |t|$.

(ii) *The point group H contains only elements of finite order, and the subgroup H_0 is cyclic with generator a rotation θ through $2\pi/q$ for some integer $q > 0$.* This follows from the fact that H, and hence also H_0, is finite; therefore every element is of finite order and H_0 contains a rotation θ through an angle $\alpha = 2\pi/q$ for which q is a maximum. If $\varphi \in H_0$ is a rotation through an angle $\alpha j + \beta$ (where j is an integer and $0 \leqslant \beta < \alpha$) then the fact that q is a maximum implies that $\beta = 0$ and $\varphi = \theta^j$. Therefore H_0 is cyclic with generator θ.

(iii) *The subgroup* H_0 *is of order* $q = 1, 2, 3, 4$ *or* 6*. If* $q = 3, 4$ *or* 6 *there is a vector* t *such that*

$$T = \{n_1 t + n_2 \theta t : n_1, n_2 \text{ integers}\}$$

where θ *is a rotation through* $2\pi/q$*.* These facts, which essentially describe the action of H on T, can be proved by considering a non-zero vector $t \in T$ of minimum length, as in (i), and the generator θ of H_0, as in (ii). We use the fact that the angle between two non-zero vectors t_1, t_2 of minimum length must be at least $\frac{1}{3}\pi$ (if the angle were less, then $t_1 - t_2$ would be a shorter vector). If $q = 2i + 1$, take the two vectors to be $-t$ and $\theta^i t$, so that $q \leqslant 3$. If $q = 2i$, take the two vectors to be t and θt, so that $q \leqslant 6$. This proves that $q = 1, 2, 3, 4$ or 6 (a fact known to physicists as the "crystallographic restriction").

The vector $n_1 t + n_2 \theta t$ is always in T. If not every vector of T has this form, and if $q \geqslant 2$, then there must exist real numbers x_1, x_2 with $0 < x_1, x_2 \leqslant \frac{1}{2}$ for which $x_1 t + x_2 \theta t \in T$. Such a vector would have length

$$(x_1^2 + x_2^2 + 2x_1x_2\cos(2\pi/q))^{\frac{1}{2}}|t| \leqslant (\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{2}\cos(2\pi/q))^{\frac{1}{2}}|t|,$$

which is $\leqslant |t|$ with equality only if $q = 2$ and $x_1 = x_2 = \frac{1}{2}$. If $q = 3, 4$ or 6 this contradicts the fact that t is of minimum length.

(iv) The shift vectors of $\varphi \in H$ are easily described, since every such $a \in T$ must satisfy $\varphi a = a$. If $\varphi \neq \iota$ is a rotation then this implies $a = 0$. If φ is reflection in a line l, then it implies that $a \in l$, and any two shift vectors of φ must differ by $t + \varphi t$ for some $t \in T$ (note that $t + \varphi t$ is always on l for *any* $t \in T$ because $\varphi(t + \varphi t) = t + \varphi t$). To describe the various situations which can arise, consider a non-zero vector $r \in T$ which lies on l and which, among such, is of minimum length: either r *is* of the form $t + \varphi t$ with $t \in T$ (we call this *situation 1*) or it is not (*situation 2*). The latter breaks up into two, giving the following possibilities:

Situation 1. In this case any two vectors of T on l differ by a vector of the form $t + \varphi t$. Therefore every $a \in T$ on l is a shift vector of φ (note that this is always the situation if $q = 3$ (take $t = -\theta r$) or if $q = 6$ (take $t = \theta r$)).

Situation 2. Now $r + \varphi r = 2r$, so that any two shift vectors of φ differ by an even multiple of the vector r. There are two further possibilities according as the shift vectors are even or odd, so that this situation breaks up into:

Situation 2(m). We can choose v so that $a = 0$.

Situation 2(g). We can choose v so that $a = r$.

In crystallography the three situations 1, 2(m), 2(g) are described as centred (cm), primitive with mirror (pm) and primitive with glide (pg) respectively. However, when $q = 3$ or 6 it is also permissible to refer to situation 1 as primitive.

3. Equivalence of plane groups

Before proving the classification we must state precisely when two plane groups are to be regarded as equivalent. The remarks at the beginning of Section 2 apply to this definition also.

DEFINITION. Two plane groups G, G' with lattices T, T' are *equivalent* if there is an isomorphism $G \to G'$ which maps the subgroup T onto T'.

Once again we examine the consequences of this definition for the four invariants defined in Section 1. These consequences may be summarised by saying that the definition of equivalence implies a relationship between the invariants of G and the invariants of G' which is so specific that it shows clearly how to prove conversely that, if the invariants of G and G' are related, then G and G' are equivalent.

(i) The restriction of an isomorphism $G \to G'$ to the subgroup T of G is a homomorphism which is both one–one and onto. Therefore it gives an isomorphism $\lambda : T \to T'$. But T and T' each contain a pair of linearly independent vectors, so that there are corresponding linear transformations λ and λ^{-1} of $\mathbb{R}^2$.

(ii) If $(v, \varphi) \in G$ is mapped to $(v', \varphi') \in G'$ then

$$(\varphi t, \iota) = (v, \varphi) . (t, \iota) . (v, \varphi)^{-1}$$

must be mapped to $(v', \varphi') . (\lambda t, \iota) . (v', \varphi')^{-1} = (\varphi' \lambda t, \iota)$, and therefore $\lambda \varphi t = \varphi' \lambda t$ for all $t \in T$. But T contains a pair of linearly independent vectors and therefore $\varphi' = \lambda \varphi \lambda^{-1}$. We conclude that the point groups H, H' of G, G' are related by $H' = \lambda H \lambda^{-1}$; the subgroups H_0, H_0' of all rotations in H, H' similarly satisfy $H_0' = \lambda H_0 \lambda^{-1}$.

(iii) The same argument shows that the action of $\varphi' = \lambda \varphi \lambda^{-1}$ on $T' = \lambda T$ is defined by composition with the action of φ on T.

(iv) If φ is of order q then so is $\varphi' = \lambda \varphi \lambda^{-1}$. Let $(v, \varphi) \in G$ map to $(v', \varphi') \in G'$. Thus $(a, \iota) = (v, \varphi)^q$ implies $(\lambda a, \iota) = (v', \varphi')^q$, and if $a \in T$ is a shift vector of φ then $\lambda a \in T'$ is a shift vector of φ'.

4. Classification theorems

The 17 equivalence classes of plane groups may be conveniently obtained in three stages: there are 5 for which the point group contains no reflections, 3 for which the point group contains a single reflection, 9 for which the point group contains more than one reflection. The method is to prove that the invariants (i), (ii), (iii), (iv) determine the equivalence class of G uniquely.

THEOREM. *There are 5 equivalence classes of plane group G whose point group contains no reflections.*

PROOF. In this case, (ii) and (iii) of Section 2 show that the point group H is cyclic with generator a rotation θ through $2\pi/q$ where $q = 1, 2, 3, 4$ or 6.

Let G and G' be two plane groups with the same point group. To show that they are equivalent, we first construct an isomorphism $\lambda : T \to T'$ such that $\theta\lambda = \lambda\theta$ (for $q = 1, 2$ any isomorphism will do; for $q = 3, 4, 6$ use (iii) of Section 2 and define $\lambda t = t'$ and $\lambda\theta^i t = \theta^i t'$, where t, t' are non-zero vectors of minimum length in T, T'). Then λ defines a linear transformation of $\mathbb{R}^2$ such that $\lambda\theta^i = \theta^i\lambda$ for $i = 0, 1, \ldots, q-1$. Write each of the groups G, G' as a union

$$G = T.(0,\iota) \cup T.(v,\theta) \cup \ldots \cup T.(v,\theta)^{q-1}$$
$$G' = T'.(0,\iota) \cup T'.(v',\theta) \cup \ldots \cup T'.(v',\theta)^{q-1}$$

of q cosets. Now define $G \to G'$ by sending $(t,\iota).(v,\theta)^i$ to $(\lambda t,\iota).(v',\theta)^i$. The fact that this is a homomorphism follows from the equations

$$(v,\theta)^i.(t,\iota) = (\theta^i t,\iota).(v,\theta)^i,$$
$$(v',\theta)^i.(\lambda t,\iota) = (\theta^i \lambda t,\iota).(v',\theta)^i = (\lambda\theta^i t,\iota).(v',\theta)^i,$$

and the fact that, by (iv) of Section 2, the shift vectors $a \in T$, $a' \in T'$ of θ are zero. Therefore $G \to G'$ is a homomorphism. Since it obviously has an inverse and maps T onto T', the groups G and G' are equivalent. We therefore have one equivalence class for each value of q; in crystallography the 5 classes are denoted p1, p2, p3, p4, p6.

THEOREM. *There are 3 equivalence classes of plane group G whose point group contains a single reflection.*

PROOF. In this case the point group H of G is of order 2 with generator the reflection ρ in a line l. As in (iv) of Section 2, there is a non-zero vector $r \in T$ which lies on l and among such is of minimum length. A similar trick shows that there is a non-zero vector $s \in T$ which is perpendicular to l and among such is of minimum length (the trick is to observe that for *any* $t \in T$ the vector $t - \rho t$ is perpendicular to l because $\rho(t - \rho t) = -(t - \rho t)$). The three situations become:

Situation 1. There is $t \in T$ such that $t + \rho t = r$; it must necessarily be possible to choose $t = \frac{1}{2}r + \frac{1}{2}s$, so that the pair r, t can be chosen as the basis for the (centred rectangular) lattice T.

Situation 2(m). The pair of vectors r, s can be chosen as the basis for the (primitive rectangular) lattice T and there is $v \in \mathbb{R}^2$ such that $(v, \rho) \in G$ and $a = v + \rho v = 0$.

Situation 2(g). The pair of vectors r, s can be chosen as the basis for the (primitive rectangular) lattice T and there is $v \in \mathbb{R}^2$ such that $(v, \rho) \in G$ and $a = v + \rho v = r$.

Now let G' be a plane group which yields the same of the above three situations as G. Thus G' has lattice T' and point group H' generated by a reflection ρ' in a line l'. Construct $r', s', a' \in T'$ as above, and define λ by

$\lambda r = r'$, $\lambda s = s'$. Then $\rho'\lambda = \lambda\rho$ and, since both groups yield the same situation, $a' = \lambda a$. Write each group as a union

$$G = T.(0,\iota) \cup T.(v,\rho)$$
$$G' = T'.(0,\iota) \cup T'.(v',\rho')$$

of two cosets, and define $G \to G'$ by sending $(t,\iota).(v,\rho)^i$ to $(\lambda t,\iota).(v',\rho')^i$ for $i = 0, 1$. That this is a homomorphism follows from the equations

$$(v,\rho).(t,\iota) = (\rho t,\iota).(v,\rho), \qquad (v,\rho)^2 = (a,\iota),$$
$$(v',\rho').(\lambda t,\iota) = (\lambda\rho t,\iota).(v',\rho'), \qquad (v',\rho')^2 = (a',\iota).$$

Since it obviously has an inverse and maps T onto T', the groups G and G' are equivalent. The three situations yield three equivalence classes which are denoted cm, pm, pg respectively.

THEOREM. *There are 9 equivalence classes of plane group G whose point group contains more than one reflection.*

PROOF. If a point group H contains two reflections ρ, σ in lines l, m then it also contains the product $\rho\sigma$ which is a rotation. We may therefore choose generators ρ, σ for H such that $\theta = \rho\sigma$ is the rotation through $2\pi/q$ which generates the group H_0 of all rotations in H. Then the lines l, m make an angle π/q, and the reflections ρ, σ determine shift vectors $a = v + \rho v$ on l and $b = w + \sigma w$ on m. The proof consists in showing that there are 9 possible combinations of point groups and shift vectors; that each combination yields a single equivalence class is proved as in the previous theorems. Let r, s be non-zero vectors in T which lie on l, m and which among such are of minimum length. If $q = 2$, then as in the previous theorem either $\frac{1}{2}r + \frac{1}{2}s \in T$ (in which case *both* ρ and σ yield situation 1) or r, s gives a basis for T (in which case there are three possibilities: both ρ and σ yield situation 2(m), or both yield 2(g), or one yields 2(m) and one yields 2(g)). Note that in the foregoing discussion r, s are interchangeable and cannot be distinguished by any property of G, so that we have 4 possible combinations of invariants when $q = 2$. If $q = 3$, then a similar argument implies that either $\frac{1}{3}r + \frac{1}{3}s \in T$ or r, s is a basis for T, giving 2 combinations (both situation 1). If $q = 4$ or 6, then r, s must necessarily form a basis and one of them (without loss of generality say r) is a non-zero vector of minimum length so that $s = r + \theta r = r + \sigma r$. If $q = 4$ then situations 2(m) and 2(g) are the only possibilities for ρ, while situation 1 is the only possibility for σ; if $q = 6$ then only situation 1 can occur. In all we therefore have $4 + 2 + 2 + 1 = 9$ combinations which are summarised in the table overleaf.

To complete the proof, suppose that G and G' are two plane groups which determine lattices T and T' with vectors $a, b, r, s \in T$ and $a', b', r', s' \in T'$ which yield the same combination in the table. Let λ be the linear transformation defined by $\lambda r = r'$, $\lambda s = s'$. Then $\rho'\lambda = \lambda\rho$, $\sigma'\lambda = \lambda\sigma$ and

$a' = \lambda a$, $b' = \lambda b$. These equations imply, by the same argument as in the previous theorems, that there is an isomorphism $G \to G'$ which sends the generator (t, ι) to $(\lambda t, \iota)$, the generator (v, ρ) to (v', ρ'), and the generator (w, σ) to (w', σ'). We conclude that the plane groups G and G' are equivalent.

q	Properties of T	Shift vectors	Notation
2	$\frac{1}{2}r + \frac{1}{2}s \in T$	$a = b = 0$	cmm
	r, s form basis for T, i.e. every $t \in T$ has the form $n_1 r + n_2 s$ (n_1, n_2 integers)	$a = b = 0$	pmm
		$a = r, b = 0$ or $a = 0, b = s$	pmg
		$a = r, b = s$	pgg
3	$\frac{1}{3}r + \frac{1}{3}s \in T$	$a = b = 0$	p31m
	r, s form basis $s = r + \theta r = r + \sigma r$		p3m1
4	r, s form basis $s = r + \theta r = r + \sigma r$	$a = b = 0$	p4mm
		$a = r, b = 0$	p4mg
6	r, s form basis $s = r + \theta r = r + \sigma r$ $r = \theta r + \rho\theta r$	$a = b = 0$	p6mm

5. *References*

The proof given above is a solution for $n = 2$ of the general problem: to find the number s_n of equivalence classes of n-dimensional space groups. The conjecture that s_n is always finite was made by Hilbert (1900) and a proof was given by Bieberbach (1910). Before this the joint efforts of Sohncke, Fedorov and Schönflies had established (1891) that $s_3 = 219$ (or, with a slightly different definition of equivalence, 230) and this description of the three-dimensional space groups can be found in most textbooks on crystallography. The fact that $s_2 = 17$ is of course implicit in this description; it is also to be found in the work of Fricke and Klein (1897) on automorphic functions. It was stated explicitly by Polya and Niggli (1924) who gave detailed information and pictures of patterns for each of the 17 groups, but did not publish a proof of the classification theorem. A full description of all the work on two-dimensional and three-dimensional space groups is included in:

J. J. Burckhardt, *Die Bewegungsgruppen der Kristallographie*. Birkhäuser (Basel, 1966).

The only other value of s_n which has been calculated to date is $s_4 = 4783$, and a description of some of the steps leading to this result can be found in:

R. Bülow, J. Neubüser and H. Wondratschek, *Acta crystallogr.* **27A**, 517–535 (1971).

Two works which are readily available, and which place the symmetry problems discussed above in a wider context are:

H. Weyl, *Symmetry*. Princeton University Press (1952).

M. J. Buerger, *Elementary crystallography*. Wiley (New York, 1956).

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Rooks inviolate

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One of the oldest and best known of chessboard problems is to place the largest possible number of similar pieces on the board such that no two of these pieces are attacking each other. In his book *Amusements in mathematics* [1], Dudeney considered this problem on a generalised square chessboard containing n^2 cells, and proved that for rooks, queens and bishops this maximum number is equal to n, n and $2n-2$ respectively.

A related problem is to find out the number of different ways in which the pieces may be arranged on the board, subject to the 'non-attacking condition'. In the case of rooks, it is necessary to have exactly one rook in each row and in each column, and it is not difficult to see that the number of ways of achieving this is $n!$. If, however, we insist that two arrangements are essentially the same whenever one can be transformed into the other by a rotation or reflection of the board, then the problem becomes much more difficult, and was solved by Dudeney only for small values of n. The corresponding problems for queens and bishops are slightly easier, and are discussed at length in [2] and [3].

In this paper the problem will be solved, in the rook case, by an application of a result from the theory of groups of transformations. Although the proof of this result (the lemma below) is rather technical, it has applications to a wide range of combinatorial problems. The present case is a good